Complexity of Manipulative Actions When Voting with Ties

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Abstract. Most of the computational study of election problems has assumed that each voter's preferences are, or should be extended to, a total order. However in practice voters may have preferences with ties. We study the complexity of manipulative actions on elections where voters can have ties, extending the definitions of the election systems (when necessary) to handle voters with ties. We show that for natural election systems allowing ties can both increase and decrease the complexity of manipulation and bribery, and we state a general result on the effect of voters with ties on the complexity of control.

1 Introduction

Elections are commonly used to reach a decision when presented with the preferences of several agents. This includes political domains as well as multiagent systems. In an election agents can have an incentive to cast a strategic vote in order to affect the outcome. An important negative result from social-choice theory, the Gibbard-Satterthwaithe theorem, states that every reasonable election system is susceptible to strategic voting (a.k.a. manipulation) [16,26].

Although every reasonable election system can be manipulated, it may be computationally infeasible to determine if a successful manipulation exists. Bartholdi et al. introduced the notion of exploring the computational complexity of the manipulation problem [1]. They expanded on this work by introducing and analyzing the complexity of control [2]. Control models the actions of an election organizer, referred to as the chair, who has control over the structure of the election (e.g., the voters) and wants to ensure that a preferred candidate wins. Faliszewski et al. introduced the model of bribery [9]. Bribery is closely related to manipulation, but instead of asking if voters can cast strategic votes to ensure a preferred outcome, bribery asks if a subcollection of the voters can be paid to change their vote to ensure a preferred outcome.

It is important that we understand the complexity of these election problems on votes that allow ties, since in practical settings voters often have ties between some of the candidates. This is seen in the online preference repository

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PrefLib, which contains several election datasets containing votes with ties, ranging from political elections to elections created from rating data [23]. Most of the computational study of election problems for partial votes has assumed that each voter's preferences should be extended to a total order (see e.g., the possible and necessary winner problems [21]). However an agent may view two options as explicitly equal and it makes sense to view these preferences as votes with ties, instead of as partial rankings that can be extended.

Election systems are sometimes even explicitly defined for voters with ties. Both the Kemeny rule [20] and the Schulze rule [27] are defined for votes that contain ties. Also, there exist variants of the Borda count that are defined for votes that contain ties [8].

The computational study of the problems of manipulation, control, and bribery has largely been restricted to elections that contain voters with tie-free votes. Important recent work by Narodytska and Walsh [25] studies the computational complexity of the manipulation problem for top orders, i.e., votes where the candidates ranked last are all tied and are otherwise total orders. The manipulation results in this paper can be seen as an extension of the work by Narodytska and Walsh. We consider orders that allow a voter to state ties at each position of his or her preference order, i.e., weak orders. We mention that in contrast to the work by Narodytska and Walsh [25], we give an example of a natural case where manipulation becomes hard when given votes with ties, while it is in P for total orders. Additionally, we are the first to study the complexity of the standard models of control and bribery for votes that contain ties. However, we mention here that Baumeister et al. consider a different version of bribery called extension bribery, for top orders (there called top-truncated votes) [3].

The organization of this paper is as follows. In Sect. 2 we state the formal definitions and problem statements needed for our results. The results in Sect. 3 are split into three sections, each showing a different behavior of voting with ties. In Sect. 3.1 we give examples of election systems where the problems of manipulation, bribery, and control increase in complexity from P to NP-complete. Conversely, in Sect. 3.2 we give examples of election systems where the complexity of manipulation and bribery becomes easier, and state a general result about the complexity of control. In Sect. 3.3 we solve an open question from Narodytska and Walsh [25] and give examples of election systems whose manipulation complexities are unaffected by voters with ties. Additionally, we completely characterize 3-candidate Copeland$^\alpha$ coalitional weighted manipulation for rational and irrational voters with ties. We discuss related work in Sect. 4 and our general conclusions and open directions in Sect. 5.

2 Preliminaries

An *election* consists of a finite set of candidates C and a collection of voters V (also referred to as a preference profile). Each voter in V is specified by its preference order. We consider voters with varying amounts of ties in their preferences. A *total order* is a linear ordering of all of the candidates from most to least preferred. A *weak order* is a transitive, reflexive, and antisymmetric ordering where the indifference relation ("$\sim$") is transitive. In general, a weak order can be viewed as a total order with ties. As usual, we will colloquially refer to indifference as ties throughout this paper since the indifference relation specifies the preference for two elements being equal. A *top order* is a weak order with all tied candidates ranked last, and a *bottom order* is a weak order with all tied candidates ranked first. In Example 1 below we present examples of each of the orders examined in this paper.

Example 1. *Given the candidate set $\{a, b, c, d\}$, $a > b \sim c > d$ is a weak order, $a \sim b > c > d$ is a bottom order, $a > b > c \sim d$ is a top order, and $a > b > c > d$ is a total order. Notice that every bottom order and every top order is also a weak order, and that every total order is also a top, bottom, and weak order.*

An *election system*, $\mathcal{E}$, maps an election, i.e., a finite candidate set C and a collection of voters V, to a set of winners, where the winner set can be any subset of the candidate set. The voters in an election can sometimes have an associated weight where a voter with weight w counts as w unweighted voters.

We examine two important families of election systems, the first being scoring rules. A scoring rule uses a vector of the form $\langle s_1, \ldots, s_m \rangle$, where m denotes the number of candidates, to determine each candidate's score when given a preference profile. When the preferences are all total orders, a candidate at position i in the preference order of a voter receives a score of s_i from that voter. The candidate(s) with the highest total score win. We consider the following three scoring rules.

Plurality: with scoring vector $\langle 1, 0, \ldots, 0 \rangle$.
Borda: with scoring vector $\langle m-1, m-2, \ldots, 1, 0 \rangle$.
t-Approval: with scoring vector $\langle \underbrace{1, \ldots, 1}_{\text{t}}, 0, \ldots, 0 \rangle$.

To properly handle voters with ties in their preference orders we define several natural extensions which generalize the extensions from Baumeister et al. [3] and Narodytska and Walsh [25].

Write a preference order with ties as $G_1 > G_2 > \cdots > G_r$ where each G_i is a set of tied candidates. For each set G_i, let $k_i = \sum_{j=1}^{i-1} \|G_j\|$ be the number of candidates strictly preferred to every candidate in the set. See the caption of Table 1 for an example.

We now introduce the following scoring-rule extensions, which as stated above, generalize previously used scoring-rule extensions [3, 25]. In Table 1 we present an example of each of these extensions for Borda.

Min: Each candidate in G_i receives a score of $s_{k_i+\|G_i\|}$.
Max: Each candidate in G_i receives a score of s_{k_i+1}.
Round down: Each candidate in G_i receives a score of s_{m-r+i}.

Average: Each candidate in G_i receives a score of

$$\frac{\sum_{j=k_i+1}^{k_i+\|G_i\|} s_j}{\|G_i\|}.$$

Table 1. The score of each candidate for preference order $a > b \sim c > d$ using Borda with each of our scoring-rule extensions. We write this order as $\{a\} > \{b, c\} > \{d\}$, i.e., $G_1 = \{a\}$, $G_2 = \{b, c\}$, and $G_3 = \{d\}$. Note that $k_1 = 0$, $k_2 = 1$, and $k_3 = 3$

Borda	score(a)	score(b)	score(c)	score(d)
Min	3	1	1	0
Max	3	2	2	0
Round down	2	1	1	0
Average	3	1.5	1.5	0

The optimistic and pessimistic models from the work by Baumeister et al. [3] are the same as our max and min extensions respectively, for top orders. All of the scoring-rule extensions for top orders found in the work by Narodytska and Walsh [25] can be realized by our definitions above, with our round-down and average extensions yielding the same scores for top orders as their round-down and average extensions. With the additional modification that $s_m = 0$ our min scoring-rule extension yields the same scores for top orders as round up in the work by Narodytska and Walsh [25].

Notice that plurality using the max scoring-rule extension for bottom orders is the same as approval voting, where each voter indicates either approval or disapproval of each candidate and the candidate(s) with the most approvals win. For example, given the set of candidates $\{a, b, c, d\}$, an approval vector that approves of a and c, and a preference order $a \sim c > b > d$ yield the same scores for approval and plurality using max respectively.

In addition to scoring rules, elections can be defined by the pairwise majority elections between the candidates. One important example is Copeland$^\alpha$ [7] (where α is a rational number between 0 and 1), which is scored as follows. Each candidate receives one point for each pairwise majority election he or she wins and receives α points for each tie. We also mention that Copeland^1 is often referred to, and will be throughout this paper, as Llull [17]. We apply the definition of Copeland$^\alpha$ to weak orders in the obvious way (as was done for top orders in [3, 25]).

We sometimes look at voters whose preferences need not be rational and we refer to those voters as "irrational." This simply means that for every unordered pair a, b of distinct candidates, the voter has $a > b$ or $b > a$. For example, a voter's preferences could be $(a > b, b > c, c > a)$. We also look at irrational votes with ties.

When discussing elections defined by pairwise majority elections we sometimes refer to the *induced majority graph* of a preference profile. A preference profile V where each voter has preferences over the set of candidates C induces the majority graph with a vertex set equal to the candidate set and an edge set defined as follows. For every $a, b \in C$ the graph contains the edge $a \rightarrow b$ if more voters have $a > b$ than $b > a$.

2.1 Election Problems

We examine the complexity of the following election problems.

The coalitional manipulation problem (where a coalition of manipulators seeks to change the outcome of the election) for weighted voters, first studied by Conitzer et al. [6], is described below.

Name: $\mathcal{E}$-CWCM
Given: A candidate set C, a collection of nonmanipulative voters V where each voter has a positive integral weight, a preferred candidate $p \in C$, and a collection of manipulative voters W.
Question: Is there a way to set the votes of the manipulators such that p is an $\mathcal{E}$ winner of the election $(C, V \cup W)$?

Electoral control is the problem of determining if it is possible for an election organizer with control over the structure of an election, whom we refer to as the election chair, to ensure that a preferred candidate wins [2]. We formally define the specific control action of constructive control by adding voters (CCAV) below. CCAV is one of the most natural cases of electoral control and it models scenarios such as targeted voter registration drives where voters whose votes will ensure the goal of the chair are added to the election.

Name: $\mathcal{E}$-CCAV
Given: A candidate set C, a collection of voters V, a collection of unregistered voters U, a preferred candidate $p \in C$, and an add limit $k \in \mathbb{N}$.
Question: Is there a subcollection of the unregistered voters $U' \subseteq U$ such that $\|U'\| \leq k$ and p is an $\mathcal{E}$ winner of the election $(C, V \cup U')$?

Bribery is the problem of determining if it is possible to change the votes of a subcollection of the voters, within a certain budget, to ensure that a preferred candidate wins [9]. The case for unweighted voters is defined below, but we also consider the case for weighted voters.

Name: $\mathcal{E}$-Bribery
Given: A candidate set C, a collection of voters V, a preferred candidate $p \in C$, and a bribe limit $k \in \mathbb{N}$.
Question: Is there a way to change the votes of at most k of the voters in V so that p is an $\mathcal{E}$ winner?

2.2 Computational Complexity

We use the following NP-complete problems in our proofs of NP-completeness.

Name: Exact Cover by 3-Sets
Given: A nonempty set of elements $B = \{b_1, \ldots, b_{3k}\}$ and a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of 3-element subsets of B.
Question: Does there exist a subcollection $\mathcal{S}'$ of $\mathcal{S}$ such that every element of B occurs in exactly one member of $\mathcal{S}'$?

Name: Partition
Given: A nonempty set of positive integers $k_1, \ldots, k_t$ such that $\sum_{i=1}^{t} k_i = 2K$.
Question: Does there exist a subset A of $k_1, \ldots, k_t$ such that $\sum A = K$?[1]

Some of our results utilize the following variation of Partition, referred to as Partition′, for which we prove NP-completeness by a reduction from Partition.

Name: Partition′
Given: A nonempty set of positive even integers $k_1, \ldots, k_t$ and a positive even integer $\widehat{K}$.
Question: Does there exist a partition (A, B, C) of $k_1, \ldots, k_t$ such that $\sum A = \sum B + \widehat{K}$?

Theorem 1. *Partition′ is* NP*-complete.*

Proof. The construction here is similar to the first part of the reduction to a different version of Partition from Faliszewski et al. [9].

Given $k_1, \ldots, k_t$ such that $\sum_{i=1}^{t} k_i = 2K$, corresponding to an instance of Partition, we construct the following instance $k'_1, \ldots, k'_t, \ell'_1, \ldots, \ell'_t, \widehat{K}$ of Partition′. Let $k'_i = 4^i + 4^{t+1} k_i$, $\ell'_i = 4^i$, and $\widehat{K} = 4^{t+1}K + \sum_{i=1}^{t} 4^i$. (Note that in Faliszewski et al. [9] "3"s were used, but we use "4"s here so that when we add a subset of $k'_1, \ldots, k'_t, \ell'_1, \ldots, \ell'_t, \widehat{K}$, we never have carries in the last $t+1$ digits base 4, and we set the last digit to 0 to ensure that all numbers are even.)

If there exists a partition (A, B, C) of $k'_1, \ldots, k'_t, \ell'_1, \ldots, \ell'_t$ such that $\sum A = \sum B + \widehat{K}$, then $\forall i, 1 \le i \le t$, $\lfloor (\sum A)/4^i \rfloor \mod 4 = \lfloor (\sum B + \widehat{K})/4^i \rfloor \mod 4$. Note that $\lfloor (\sum A)/4^i \rfloor \mod 4 = \|A \cap \{k'_i, \ell'_i\}\|$, $\lfloor (\sum B)/4^i \rfloor \mod 4 = \|B \cap \{k'_i, \ell'_i\}\|$, and $\lfloor \widehat{K}/4^i \rfloor \mod 4 = 1$. So, $\|A \cap \{k'_i, \ell'_i\}\| = \|B \cap \{k'_i, \ell'_i\}\| + 1$. It follows that exactly one of k'_i or ℓ'_i is in A and neither is in B. Since this is the case for every i, it follows that $B = \emptyset$. Now look at all k_i such that k'_i is in A. That set will add up to K, and so our original Partition instance is a positive instance.

For the converse, it is immediate that a subset D of $k_1, \ldots, k_t$ that adds up to K can be converted into a solution for our Partition′ instance, namely, by putting k'_i in A for every k_i in D, putting ℓ'_i in A for every k_i not in D, letting $B = \emptyset$, and putting all other elements of $k'_1, \ldots, k'_t, \ell'_1, \ldots, \ell'_t$ in C. □

3 Results

3.1 Complexity Goes up

The related work on the complexity of manipulation of top orders [25] did not find a natural case where manipulation complexity increases when moving from total orders to top orders. We will show such cases in this section.

[1] Here and elsewhere we write $\sum A$ to denote $\sum_{a \in A} a$.

Single-peakedness is a restriction on the preferences of the voters introduced by Black [4]. Given a total order A over the candidates, referred to as an axis, a collection of voters is single-peaked with respect to A if each voter has preferences that strictly increase to a peak and then strictly decrease, only strictly increase, or only strictly decrease with respect to A.

For our purposes we consider the model of top order single-peakedness introduced by Lackner [22] where given an axis A, a collection of voters is single-peaked with respect to A if no voter has preferences that strictly decrease and then strictly increase with respect to A. Notice that for total orders, if a preference profile is single-peaked with respect to Black's model [4] it is also single-peaked with respect to Lackner's model [22].

For single-peaked preferences we follow the model of manipulation from Walsh [28] where the axis is given and both the nonmanipulators and the manipulators all cast votes that are single-peaked with respect to the given axis. 3-candidate Borda CWCM is known to be in P for single-peaked voters [12].

Theorem 2. *[12] 3-candidate Borda CWCM for single-peaked total orders is in* P*.*

We now consider the complexity of 3-candidate Borda CWCM for top orders that are single-peaked. In all of our reductions the axis is $a <_A p <_A b$. Single-peakedness with respect to this axis allows the following top order votes: $a > p > b$, $a \sim p \sim b$, $a > p \sim b$, $p > a > b$, $p > b > a$, $p > a \sim b$, $b > p > a$, and $b > p \sim a$. It does not allow $a > b > p$ or $b > a > p$.

Theorem 3. *3-candidate Borda CWCM for single-peaked top orders using max is* NP*-complete.*

Proof. Given a nonempty set of positive integers $k_1, \ldots, k_t$ such that $\sum_{i=1}^{t} k_i = 2K$ we construct the following instance of manipulation.

Let the set of candidates be $C = \{a, b, p\}$. We have two nonmanipulators with the following weights and votes.

- One weight $3K$ nonmanipulator voting $a > p \sim b$.
- One weight $3K$ nonmanipulator voting $b > p \sim a$.

From the nonmanipulators, score(p) $= 6K$, while score(a) and score(b) are both $9K$.

Let there be t manipulators, with weights $k_1, \ldots, k_t$. Without loss of generality, all of the manipulators put p first. Then p receives a score of $10K$ overall. However, a and b can score at most K each from the votes of the manipulators, for p to be a winner. So the manipulators must split their votes so that a subcollection of manipulators with weight K votes $p > a > b$ and a subcollection with weight K votes $p > b > a$. Notice that these are the only votes possible to ensure that p wins and that the manipulators cannot simply all vote $p > a \sim b$ since both a and b receive a point from that vote (since we are using max) and we have no points to spare. $\square$

The above argument for max does not immediately apply to the other scoring-rule extensions. In particular, for min the optimal vote for the manipulators is always to rank p first and to rank the remaining candidates tied and less preferred than p (as in Proposition 3 of Narodytska and Walsh [25]). So that case is in P, with an optimal manipulator vote of $p > a \sim b$.

It is not hard to modify the proof to show that the reduction of the proof of Theorem 3 also works for the round-down case.

Theorem 4. *3-candidate Borda CWCM for single-peaked top orders using round down is* NP*-complete.*

The average scoring-rule extension case is more complicated since it is less close to Partition than the previous cases. We will still be able to show NP-completeness, but we have to reduce from the special, restricted version of Partition that we defined previously in Sect. 2.2 as Partition′.[2]

Theorem 5. *3-candidate Borda CWCM for single-peaked top orders using average is* NP*-complete.*

Proof. Let $k_1, \ldots, k_t, \widehat{K}$ be an instance of Partition′. We are asking whether there exists a partition (A, B, C) of $k_1, \ldots, k_t$ such that $\sum A = \sum B + \widehat{K}$. Recall that all numbers involved are even. Let $k_1, \ldots, k_t$ sum to $2K$. Without loss of generality, assume that $\widehat{K} \leq 2K$.

Let the candidates $C = \{a, b, p\}$. We have two nonmanipulators with the following weights and votes.

- One weight $6K + \widehat{K}$ nonmanipulator voting $a > p \sim b$.
- One weight $6K - \widehat{K}$ nonmanipulator voting $b > p \sim a$.

From the nonmanipulators, score(p) is $6K$, score(a) + score(b) $= 30K$ and score(a) − score(b) $= 3\widehat{K}$.

Let there be t manipulators, with weights $3k_1, \ldots, 3k_t$.

First suppose there exists a partition (A, B, C) of $k_1, \ldots, k_t$ such that $\sum A = \sum B + \widehat{K}$. For every $k_i \in A$, let the weight $3k_i$ manipulator vote $p > b > a$. For every $k_i \in B$, let the weight $3k_i$ manipulator vote $p > a > b$. For every $k_i \in C$, let the weight $3k_i$ manipulator vote $p > a \sim b$. Notice that after this manipulation that score(p) $= 18K$, score(a) = score(b), and score(a) + score(b) $= 30K + 6K$. It follows that score(p) = score(a) = score(b) $= 18K$.

For the converse, suppose that p can be made a winner. Without loss of generality, assume that p is ranked uniquely first by all manipulators. Then score(p) = score(a) = score(b) $= 18K$. Let A' be the set of manipulator weights that vote $p > b > a$, let B' be the set of manipulator weights that vote $p > a > b$, and let C' be the set of manipulator weights that vote $p > a \sim b$. No

[2] A similar situation occurred in the proof of Proposition 5 in Narodytska and Walsh [25], where a (very different) specialized version of Subset Sum was constructed to prove that 3-candidate Borda CWCM (in the non-single-peaked case) for top orders using average remained NP-complete.

other votes are possible. Let $A = \{k_i \mid 3k_i \in A'\}$, $B = \{k_i \mid 3k_i \in B'\}$, and $C = \{k_i \mid 3k_i \in C'\}$. Therefore (A, B, C) corresponds to a partition of $k_1, \dots, k_t$. Note that $\sum A = \sum B + \widehat{K}$. □

We now consider cases where the complexity of control can increase when moving from total order votes to votes with ties. We examine the complexity of CCAV, which is one of the most natural models of control and known to be in P for plurality for total orders [2].

Theorem 6. *[2] Plurality CCAV for total orders is in* P.

However below we show two cases where CCAV for plurality is NP-complete for bottom orders and weak orders.

As mentioned in the Preliminaries, plurality using max for bottom orders is the same as approval voting. So the theorem below immediately follows from the proof of Theorem 4.43 from Hemaspaandra et al. [19].

Theorem 7. *Plurality CCAV for bottom orders and weak orders using max is* NP-*complete.*

We now show that the case of plurality for bottom orders and weak orders using average is NP-complete.

Theorem 8. *Plurality CCAV for bottom orders and weak orders using average is* NP-*complete.*

Proof. Let $B = \{b_1, \dots, b_{3k}\}$ and a collection $\mathcal{S} = \{S_1, \dots S_n\}$ of 3-element subsets of B be an instance of Exact Cover by 3-Sets, where each $S_j = \{b_{j_1}, b_{j_2}, b_{j_3}\}$. Without loss of generality let k be divisible by 4 and let $\ell = 3k/4$. We construct the following instance of control by adding voters.

Let the candidates $C = \{p\} \cup B$. Let the addition limit be k. Let the collection of registered voters consist of the following $(3k^2 + 9k)/4 + 1$ voters. (When "$\cdots$" appears at the end of a vote the remaining candidates from C are ranked lexicographically. For example, given the candidate set $\{a, b, c, d\}$, the vote $b > \cdots$ denotes the vote $b > a > c > d$.)

– For each i, $1 \leq i \leq \ell$, $k + 3$ voters voting $b_i \sim b_{i+\ell} \sim b_{i+2\ell} \sim b_{i+3\ell} > \cdots$.
– One voter voting $p > \cdots$.

Let the collection of unregistered voters consist of the following n voters.

– For each $S_j \in \mathcal{S}$, one voter voting $p \sim b_{j_1} \sim b_{j_2} \sim b_{j_3} > \cdots$.

Notice that from the registered voters, the score of each b_i candidate is $(k-1)/4$ greater than the score of p. Thus the chair must add voters from the collection of unregistered voters so that no b_i candidate receives more than $1/4$ more points, while p must gain $k/4$ points. Therefore the chair must add the voters that correspond to an exact cover. □

We now present a case where the complexity of bribery goes from P for total orders to NP-complete for votes with ties.

Theorem 9. *[9] Unweighted bribery for plurality for total orders is in* P*.*

The proof that bribery for plurality for bottom orders and weak order using max is NP-complete immediately follows from the proof of Theorem 4.2 from Faliszewski et al. [9], which showed bribery for approval to be NP-complete.

Theorem 10. *Unweighted bribery for plurality for bottom orders and weak orders using max is* NP*-complete.*

3.2 Complexity Goes down

Narodytska and Walsh [25] show that the complexity of coalitional manipulation can go down when moving from total orders to top orders. In particular, they show that the complexity of coalitional manipulation (weighted or unweighted) for Borda goes from NP-complete to P for top orders using round-up. This is because in round-up an optimal manipulator vote is to put p first and have all other candidates tied for last.

In contrast, notice that the complexity of a (standard) control action cannot decrease when more lenient votes are allowed. This is because the votes that create hard instances of control are still able to be cast when more general votes are possible. The election chair is not able to directly change votes, except in a somewhat restricted way in candidate control cases, but it is clear to see how this does not affect the statement below.

Observation 11. *If a (standard) control problem is hard for a type of vote with ties, it remains hard for votes that allow more ties.*

What about bribery? Bribery can be viewed as a two-phase action consisting of control by deleting voters followed by manipulation. Hardness for a bribery problem is typically caused by hardness of the corresponding deleting voters problem or the corresponding manipulation problem. If the deleting voters problem is hard, this problem remains hard for votes that allow ties, and it is likely that the bribery problem remains hard as well. Our best chance of finding a bribery problem that is hard for total orders and easy for votes with ties is a problem whose manipulation problem is hard, but whose deleting voters problem is easy. Such problems exist, e.g., all weighted m-candidate t-approval systems except plurality and triviality.[3]

Theorem 12. *[9] Weighted bribery for m-candidate t-approval for all $t \geq 2$ and $m > t$ is* NP*-complete.*

For m-candidate t-approval elections (except plurality and triviality) the corresponding weighted manipulation problem was shown to be NP-complete by Hemaspaandra and Hemaspaandra [18] and the corresponding deleting voters problem was shown to be in P by Faliszewski et al. [10].

[3] By triviality we mean a scoring rule with a scoring vector that gives each candidate the same score.

Theorem 13. *Weighted bribery for m-candidate t-approval for weak orders and for top orders using min is in* P.

Proof sketch. To perform an optimal bribery, we cannot simply perform an optimal deleting voter action followed by an optimal manipulation action. For example, if the score of b is already at most the score of p, it does not make sense to delete a voter with vote $b > p \sim a$. But in the case of bribery, we would change this voter to $p > a \sim b$, which could be advantageous.

However, the weighted constructive control by deleting voters (WCCDV) algorithm from [10] still basically works. Since m is constant, there are only a constant number of different votes possible. And we can assume without loss of generality that we bribe only the heaviest voters of each vote-type and that each bribed voter is bribed to put p first and have all other candidates tied for last. In order to find out if there exists a successful bribery of k voters, we look at all the ways we can distribute this k among the different types of votes. We then manipulate the heaviest voters of each type to put p first and have all other candidates tied for last, and see if that makes p a winner. □

3.3 Complexity Remains the Same

Narodytska and Walsh [25] show that 4-candidate Copeland$^{0.5}$ CWCM remains NP-complete for top orders. They conjecture that this is also the case for 3 candidates and point out that the reduction that shows this for total orders from Faliszewski et al. [13] won't work. We will prove their conjecture, with a reduction similar to the proof of Theorem 5.[4]

Theorem 14. *3-candidate Copeland$^{\alpha}$ CWCM remains* NP*-complete for top orders, bottom orders, and weak orders, for all rational $\alpha \in [0, 1)$ in the nonunique winner case (our standard model).*

Proof. Let $k_1, \ldots, k_t$ and $\widehat{K}$ be an instance of Partition$'$, which asks whether there exists a partition (A, B, C) of $k_1, \ldots, k_t$ such that $\sum A = \sum B + \widehat{K}$.

Let $k_1, \ldots, k_t$ sum to $2K$ and without loss of generality assume that $\widehat{K} \leq 2K$. We now construct an instance of CWCM. Let the candidate set $C = \{a, b, p\}$ and let the preferred candidate be p. Let there be two nonmanipulators with the following weights and votes.

- One weight $K + \widehat{K}/2$ nonmanipulator voting $a > b > p$.
- One weight $K - \widehat{K}/2$ nonmanipulator voting $b > a > p$.

From the votes of the nonmanipulators, score$(a) = 2$, score$(b) = 1$, and score$(p) = 0$. In the induced majority graph, there is the edge $a \rightarrow b$ with weight $\widehat{K}$, the edge $a \rightarrow p$ with weight $2K$, and the edge $b \rightarrow p$ with weight $2K$. Let there be t manipulators with, weights $k_1, \ldots, k_t$.

[4] Menon and Larson independently proved the top order case of the following theorem [24].

Suppose that there exists a partition of $k_1, \dots, k_t$ into (A, B, C) such that $\sum A = \sum B + \widehat{K}$. Then for each $k_i \in A$, have the manipulator with weight k_i vote $p > b > a$, for each $k_i \in B$, have the manipulator with weight k_i vote $p > a > b$, and for each $k_i \in C$ have the manipulator with weight k_i vote $p > a \sim b$. From the votes of the nonmanipulators and manipulators, score(a) = score(b) = score$(p) = 2\alpha$.

For the other direction, suppose that p can be made a winner. When all of the manipulators put p first then score$(p) = 2\alpha$ (the highest score that p can achieve). Since $\alpha < 1$, the manipulators must have voted such that a and b tie. This means that a subcollection of the manipulators with weight K voted $p > b > a$, a subcollection with weight $K - \widehat{K}$ voted $p > a > b$, and a subcollection with weight $\widehat{K}$ voted $p > a \sim b$. No other votes would cause b and a to tie. Notice that the weights of the manipulators in the three different subcollections form a partition (A, B, C) of $k_1, \dots, k_t$ such that $\sum A = \sum B + \widehat{K}$. $\square$

3-candidate Copeland$^\alpha$ CWCM is unusual in that the complexity can be different if we look at the unique winner case instead of the nonunique winner case (our standard model). We can prove that the only 3-candidate Copeland CWCM case that is hard for the unique winner model remains hard using a very similar approach.

Theorem 15. *3-candidate Copeland^0 CWCM remains* NP*-complete for top orders, bottom orders, and weak orders, in the unique winner case.*

Proof. Let $k_1, \dots, k_t$ and $\widehat{K}$ be an instance of Partition$'$, which asks whether there exists a partition (A, B, C) of $k_1, \dots, k_t$ such that $\sum A = \sum B + \widehat{K}$.

Let $k_1, \dots k_t$ sum to $2K$ and without loss of generality assume that $\widehat{K} \leq 2K$. We now construct an instance of CWCM. Let the candidate set $C = \{a, b, p\}$. Let the preferred candidate be $p \in C$. Let there be two nonmanipulators with the following weights and votes.

- One weight $K + \widehat{K}/2$ nonmanipulator voting $a > p > b$.
- One weight $K - \widehat{K}/2$ nonmanipulator voting $b > a > p$.

From the votes of the nonmanipulators score$(a) = 2$, score$(b) = 0$, and score$(p) = 1$. The induced majority graph contains the edge $a \to b$ with weight $\widehat{K}$, the edge $a \to p$ with weight $2K$, and the edge $p \to b$ with weight $\widehat{K}$. Let there be t manipulators, with weights $k_1, \dots, k_t$.

Suppose that there exists a partition of $k_1, \dots, k_t$ into (A, B, C) such that $\sum A = \sum B + \widehat{K}$. Then for each $k_i \in A$ have the manipulator with weight k_i vote $p > b > a$, for each $k_i \in B$ have the manipulator with weight k_i vote $p > a > b$, and for each $k_i \in C$ have the manipulator with weight k_i vote $p > a \sim b$. From the votes of the nonmanipulators and the manipulators score$(p) = 1$ and score(a) = score$(b) = 0$.

For the other direction, suppose that p can be made a unique winner. When all of the manipulators put p first then score$(p) = 1$. So the manipulators must have voted so that a and b tie, since otherwise either a or b would tie with p and

p would not be a unique winner. Therefore a subcollection of the manipulators with weight K voted $p > b > a$, a subcollection with weight $K - \widehat{K}$ voted $p > a > b$, and a subcollection with weight $\widehat{K}$ voted $p > a \sim b$. No other votes would cause a and b to tie. □

Theorem 16. *3-candidate Copeland$^\alpha$ CWCM remains in* P *for top orders, bottom orders, and weak orders, for $\alpha = 1$ for the nonunique winner case and for all rational $\alpha \in (0, 1]$ in the unique winner case.*

The proof of this theorem follows using the same arguments as the proof of the case without ties from Faliszewski et al. [13].

Tournament Result. We now state a general theorem on two-voter tournaments for votes with ties. See Brandt et al. [5] for related work on tournaments constructed from a fixed number of voters with total orders.

Theorem 17. *A majority graph can be induced by two weak orders if and only if it can be induced by two total orders.*

Proof sketch. Given two weak orders v_1 and v_2 that describe preferences over a candidate set C, we construct two total orders, v_1' and v_2' iteratively as follows.

For each pair of candidates $a, b \in C$ and $i \in \{1, 2\}$, if $a > b$ in v_i then set $a > b$ in v_i'.

For each pair of candidates $a, b \in C$, if $a > b$ in v_1 (v_2) and $a \sim b$ in v_2 (v_1) then the majority graph induced by v_1 and v_2 contains the edge $a \rightarrow b$. To ensure that the majority graph induced by v_1' and v_2' contains the edge $a \rightarrow b$ we must set $a > b$ in v_2' (v_1').

After performing the above steps there may still be a set of candidates $C' \subseteq C$ such that v_1 and v_2 are indifferent between each pair of candidates in C'. For each pair of candidates $a, b \in C'$, $a \sim b$ in v_1 and v_2, which implies the majority graph does not contain and edge between a and b. To ensure that majority graph induced by v_1' and v_2' does not contain an edge between a and b, without loss of generality set v_1' to strictly prefer the lexicographically smaller to the lexicographically larger candidate and the reverse in v_2'.

The process described above constructs two orders v_1' and v_2' and ensures that the majority graph induced by v_1 and v_2 is the same as the majority graph induced by v_1' and v_2'. Since for each pair of candidates $a, b \in C$ and $i \in \{1, 2\}$ we consider each possible case where $a \sim b$ is in v_i and set either $a > b$ or $b > a$ in the corresponding order v_i', it is clear that v_1' and v_2' are total orders. □

Observe that as a consequence of Theorem 17 we get a transfer of NP-hardness from total orders to weak orders for two manipulators when the result depends only on the induced majority graph. The proofs for Copeland$^\alpha$ unweighted manipulation for two manipulators for all rational α for total orders depend only on the induced majority graph [13,14], so we can state the following corollary to Theorem 17.

Corollary 18. *Copeland$^\alpha$ unweighted manipulation for two manipulators for all rational $\alpha \neq 0.5$ for weak orders is* NP*-complete.*

Irrational Voter Copeland Results. As mentioned in the preliminaries, another way to give more flexibility to voters is to let the voters be irrational. A voter with irrational preferences can state preferences that are not necessarily transitive and as mentioned in Faliszewski et al. [11] a voter is likely to posses preferences that are not transitive when making a decision based on multiple criteria.

Additionally, the preferences of voters can include ties as well as irrationality. When voters are able to state preferences that can contain irrationality and ties they can represent all possible pairwise preferences that they may have over all of the candidates.

It is known that unweighted Copeland$^\alpha$ manipulation is NP-complete for total orders for all rational α except 0.5 [13,14]. For irrational voters, this problem is in P for $\alpha = 0$, 0.5, and 1, and NP-complete for all other α [14]. *Weighted* manipulation for Copeland$^\alpha$ has not been studied for irrational voters. We will do so here.

Theorem 19. *3-candidate Copeland$^\alpha$ CWCM remains in* P *for irrational voters with or without ties, for $\alpha = 1$ for the nonunique winner case and for all rational $\alpha \in (0, 1]$ in the unique winner case.*

Theorem 20. *3-candidate Copeland$^\alpha$ CWCM remains* NP*-complete for irrational voters with or without ties, for $\alpha = 0$ in the unique winner case and for all rational $\alpha \in [0, 1)$ in the nonunique winner case.*

The proofs of the above two theorems follow from the arguments in the proofs of the corresponding rational cases, i.e., the proofs of Theorem 4.1 and 4.2 from Faliszewski et al. [13] for the case of voters without ties and the proofs of Theorems 14, 15, and 16 above for the case of voters with ties.

When $\alpha = 1$, also known as Llull, interesting things happen. It is known that 4-candidate Llull CWCM is in P for the unique and nonunique winner cases [15]. For larger fixed numbers of candidates, this is open. Though it is known that unweighted manipulation for Llull (with an unbounded number of candidates) is NP-complete in the nonunique winner case [14]. In contrast, we will show now that for irrational voters, all these problems are in P.

Theorem 21. *Llull CWCM is in* P *for irrational voters with or without ties, in the nonunique winner case and in the unique winner case.*

Proof. Given a set of candidates C, a collection of voters V, k manipulators, and a preferred candidate $p \in C$, the preferences of the manipulators will always contain $p > a$ for all candidates $a \neq p$. This determines the score of p. In addition, let the initial preferences of the manipulators be $a > b$ for each pair of candidates $a, b \in C - \{p\}$ such that a defeats b in V or such that a ties b in V and a is lexicographically smaller than b. Note that, if $k > 0$, there are no pairwise ties in the election with the manipulators set in this way and that the manipulators all have strict preferences between every pair of candidates (i.e., no ties in their preferences). For every $a \neq p$, let $\text{score}_0(a)$ be the score of a with the manipulators set as above.

Construct the following flow network. The nodes are: a source s, a sink t, and all candidates other than p. For every $a \in C - \{p\}$, add an edge with capacity $\text{score}_0(a)$ from s to a and add an edge with capacity $\text{score}(p)$ from a to t. For every $a, b \in C - \{p\}$, add an edge from candidate a to candidate b with capacity 1 if, when all manipulators set $b > a$, the score of a decreases by 1 (and the score of b increases by 1).

If there is manipulation such that p is a winner, then for every candidate $a \in C - \{p\}$, $\text{score}(a) \leq \text{score}(p)$ so there is a network flow that saturates all edges that go out from s.

If there is a network flow that saturates all edges that go out from s then for every $a, b \in C - \{p\}$ such that there is a unit of flow from a to b, change $a > b$ to $b > a$ in all manipulators.

This construction can be adapted to the unique winner case by letting the capacity of the edge from a to t be $\text{score}(p) - 1$ instead of $\text{score}(p)$. □

4 Related Work

The recent work by Narodytska and Walsh [25] studied the complexity of manipulation for top orders and is very influential to our computational study of more general votes with ties. Baumeister et al. [3] and Narodytska and Walsh [25] studied several extensions for election systems for top orders, which we further extend for weak orders.

Most of the related work in the computational study of election problems assumes that the partial or tied preferences of the voters must be extended to total orders. We mention the important work on partial orders by Konczak and Lang [21] that introduces the possible and necessary winner problems. Given a preference profile of partial votes, a possible winner is a candidate that wins in at least one extension of the votes to total orders, while a necessary winner wins in every extension [21].

Baumeister et al. [3] also look at the possible winner problem and in their case they examine the problem given different types of incomplete votes, i.e., top truncated, bottom truncated, and top and bottom truncated. Baumeister et al. also introduced the problem of extension bribery, where given voters with preferences that are top truncated, voters are paid to extend their vote to ensure that a preferred candidate wins [3]. We do not consider the problem of extension bribery, but instead we use the standard model of bribery introduced by Faliszewski et al. [9]. In this model the briber can set the entire preferences of a subcollection of voters to ensure that a preferred candidate wins [9].

5 Conclusions and Future Work

We examined the computational complexity of the three most commonly studied manipulative attacks on elections when voting with ties. We found a natural case for manipulation where the complexity increases for voters with ties, whereas it is easy for total orders. For bribery we found examples where the complexity increases and where it decreases. We examined the complexity of Copeland$^{\alpha}$ elections for voters with ties and even irrational votes with and without ties. It would be interesting to see how the complexity of other election problems are affected by voters with ties, specifically weak orders, which we consider to be a natural model for preferences in practical settings.

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References

1. Bartholdi III, J., Tovey, C., Trick, M.: The computational difficulty of manipulating an election. Soc. Choice Welf. **6**(3), 227–241 (1989)
2. Bartholdi III, J., Tovey, C., Trick, M.: How hard is it to control an election? Math. Comput. Model. **16**(8/9), 27–40 (1992)
3. Baumeister, D., Faliszewski, P., Lang, J., Rothe, J.: Campaigns for lazy voters: truncated ballots. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems, pp. 577–584, June 2012
4. Black, D.: On the rationale of group decision-making. J. Polit. Econ. **56**(1), 23–34 (1948)
5. Brandt, F., Harrenstein, P., Kardel, K., Seedig, H.: It only takes a few: on the hardness of voting with a constant number of agents. In: Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems, pp. 375–382, May 2013
6. Conitzer, V., Sandholm, T., Lang, J.: When are elections with few candidates hard to manipulate? J. ACM **54**(3), 1–33 (2007). Article 14
7. Copeland, A.: A "reasonable" social welfare function. Mimeographed notes from a Seminar on Applications of Mathematics to the Social Sciences, University of Michigan (1951)
8. Emerson, P.: The original Borda count and partial voting. Soc. Choice Welf. **40**(2), 352–358 (2013)
9. Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L.: How hard is bribery in elections? J. Artif. Intell. Res. **35**, 485–532 (2009)
10. Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L.: Weighted electoral control. In: Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems, pp. 367–374, May 2013
11. Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L., Rothe, J.: Llull and Copeland voting computationally resist bribery and constructive control. J. Artif. Intell. Res. **35**, 275–341 (2009)
12. Faliszewski, P., Hemaspaandra, E., Hemaspaandra, L., Rothe, J.: The shield that never was: societies with single-peaked preferences are more open to manipulation and control. Inf. Comput. **209**(2), 89–107 (2011)
13. Faliszewski, P., Hemaspaandra, E., Schnoor, H.: Copeland voting: ties matter. In: Proceedings of the 7th International Conference on Autonomous Agents and Multiagent Systems, pp. 983–990, May 2008

14. Faliszewski, P., Hemaspaandra, E., Schnoor, H.: Manipulation of Copeland elections. In: Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems, pp. 367–374, May 2010
15. Faliszewski, P., Hemaspaandra, E., Schnoor, H.: Weighted manipulation for four-candidate Llull is easy. In: Proceedings of the 20th European Conference on Artificial Intelligence, pp. 318–323, August 2012
16. Gibbard, A.: Manipulation of voting schemes. Econometrica **41**(4), 587–601 (1973)
17. Hägele, G., Pukelsheim, F.: The electoral writings of Ramon Llull. Stud. Lulliana **41**(97), 3–38 (2001)
18. Hemaspaandra, E., Hemaspaandra, L.: Dichotomy for voting systems. J. Comput. Syst. Sci. **73**(1), 73–83 (2007)
19. Hemaspaandra, E., Hemaspaandra, L., Rothe, J.: Anyone but him: the complexity of precluding an alternative. Artif. Intell. **171**(5–6), 255–285 (2007)
20. Kemeny, J.: Mathematics without numbers. Daedalus **88**, 577–591 (1959)
21. Konczak, K., Lang, J.: Voting procedures with incomplete preferences. In: Proceedings of the IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling, pp. 124–129, July/August 2005
22. Lackner, M.: Incomplete preferences in single-peaked electorates. In: Proceedings of the 28th AAAI Conference on Artificial Intelligence, pp. 742–748, July 2014
23. Mattei, N., Walsh, T.: PrefLib: a library for preferences. In: Perny, P., Pirlot, M., Tsoukiàs, A. (eds.) ADT 2013. LNCS, vol. 8176, pp. 259–270. Springer, Heidelberg (2013)
24. Menon, V., Larson, K.: Complexity of manipulation in elections with partial votes. Technical report. arXiv:1505.05900 [cs.GT], arXiv.org, May 2015
25. Narodytska, N., Walsh, T.: The computational impact of partial votes on strategic voting. In: Proceedings of the 21st European Conference on Artificial Intelligence, pp. 657–662, August 2014
26. Satterthwaite, M.: Strategy-proofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J. Econ. Theor. **10**(2), 187–217 (1975)
27. Schulze, M.: A new monotonic and clone-independent, reversal symmetric, and condorcet-consistent single-winner election method. Soc. Choice Welf. **36**(2), 267–303 (2011)
28. Walsh, T.: Uncertainty in preference elicitation and aggregation. In: Proceedings of the 22nd AAAI Conference on Artificial Intelligence, pp. 3–8, July 2007