College of the Holy Cross, Fall Semester, 2016
Math 242, Midterm 3 Practice Questions
Solutions

1. (a) Write the definition of $\lim_{x \to c} f(x) = L$.

   Solution. Let f be defined in some deleted neighborhood of c. Then $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ for every x such that $0 < |x - c| < \delta$.

   (b) Use the definition to prove that $\lim_{x \to 3} x^2 + 2x = 15$.

   Solution. First write $|x^2 + 2x - 15| = |x - 3||x + 5|$. Next let $\delta_1 = 1$. If $0 < |x - 3| < 1$, then we have $2 < x < 4$, so $7 < x + 5 < 9$ and thus $|x + 5| < 9$. Hence we have $|x - 3||x + 5| \leq 9|x - 3| < \epsilon$ when $|x - 3| < \frac{\epsilon}{9}$. So let $\delta_2 = \frac{\epsilon}{9}$ and define $\delta \min\{\delta_1, \delta_2\}$. Then, when $0 < |x - 3| < \delta$ we have $|x - 3| < 1$ and $|x - 3| < \frac{\epsilon}{9}$, so $|x + 5| < 9$ and thus

$$|x^2 + 2x - 15| = |x + 5||x - 3| \leq 9|x - 3| < 9\frac{\epsilon}{9} = \epsilon.$$

   (c) Use the definition to prove that $\lim_{x \to 8} x^{1/3} = 2$.

   Solution. First, using the equation $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ with $a = x^{1/3}$ and $b = 2$ gives $x - 8 = (x^{1/3} - 2)(x^{2/3} + 2x^{1/3} + 4)$. Now $x^{2/3} + 2x^{1/3} + 4 = (x^{1/3} + 1)^2 + 3 \geq 3$ for all x, so

$$|x^{1/3} - 2| = \frac{|x - 8|}{|x^{2/3} + 2x^{1/3} + 4|} \leq \frac{1}{3}|x - 8|$$

   for all x. Thus, given $\epsilon > 0$, we may choose $\delta = 3\epsilon$. Then whenever $0 < |x - 8| < \delta$ we have $|x^{1/3} - 2| \leq \frac{1}{3}|x - 8| < \frac{1}{3} \cdot 3\epsilon = \epsilon$.

2. Let $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 2x^2 & x \in \mathbb{Q}^c \end{cases}$

   (a) Prove that $\lim_{x \to 0} f(x) = 0$.

   Solution. Since $x^2 \leq f(x) \leq 2x^2$ for all x and $\lim_{x \to 0} x^2 = \lim_{x \to 0} 2x^2 = 0$, the Squeeze Theorem implies that $\lim_{x \to 0} f(x) = 0$.

   (b) Use appropriately chosen sequences to prove that $\lim_{x \to 1} f(x)$ does not exist.

   Solution. Let $x_n = 1 + \frac{1}{n}$ and $y_n = 1 + \frac{\sqrt{2}}{n}$. Then $x_n \neq 1$ and $y_n \neq 1$ for all n and $\lim x_n = \lim y_n = 1$. Since $x_n \in \mathbb{Q}$, $\lim f(x_n) = \lim x_n^2 = 1$, and since $y_n \in \mathbb{Q}^c$, $\lim f(y_n) = 2y_n^2 = 2$. Hence $\lim_{x \to 1} f(x)$ does not exist.

3. True/False. Prove your assertions.

   (a) If f is bounded and continuous on $(0, 1)$, then f attains a maximum value on $(0, 1)$.

   Solution. False. A counter-example is $f(x) = \frac{1}{x}$. f is continuous on $(0, 1)$ but $\lim_{x \to 0^+} f(x) = +\infty$ so f does not attain a maximum value on $(0, 1)$.

   (b) If $\lim_{x \to 5} f(x) = 0.3$, then there exists some $\delta > 0$ such that $f(x) > 0.28$ whenever $0 < |x - 5| < \delta$.

   Solution. True. Let $\epsilon = 0.02$. Then there exists some $\delta > 0$ such that $|f(x) - 0.3| < 0.02$ whenever $0 < |x - 5| < \delta$. The inequality $|f(x) - 0.3| < 0.02$ implies $-0.02 < f(x) - 0.3 < 0.02$, so $0.28 < f(x) < 0.32$. Hence $f(x) > 0.28$ whenever $0 < |x - 5| < \delta$.

(c) If $f(0) = 2$ and $f(3) = 5$, then there exists some $c \in (0, 3)$ such that $f(c) = 3$.
Solution. False. We were not told that f is continuous, so IVT does not necessarily apply. A counter-example is the function f defined piecewise by $f(x) = 2$ if $x \leq 1$ and $f(x) = 5$ if $x > 1$.

(d) The function $f(x) = \frac{\sin(x^2+1)}{2+\cos(3x)}$ attains a maximum value on $[-50, 50]$.
Solution. True. Since $2 + \cos(3x) \geq 1$ for all x, the function f is continuous on $[-50, 50]$ so by the Extreme Value Theorem, it attains a maximum value on $[-50, 50]$.

(e) If $\lim f(2 + 1/n) = -3$, then $\lim_{x\to 2} f(x) = -3$.
Solution. False. Let f be defined piecewise by $f(x) = -3$ if $x > 2$ and $f(x) = 4$ if $x \leq 2$. Then $\lim f(2 + \frac{1}{n}) = \lim -3 = -3$ but $\lim f(2 - \frac{1}{n}) = \lim 4 = 4$ so $\lim_{x\to 2} f(x)$ does not exist.

(f) If $\lim_{x\to 2} f(x) = -3$, then $\lim f(2 + 1/n) = -3$.
Solution. True. Since $\lim_{x\to 2} f(x) = -3$, if x_n is any sequence such that $x_n \neq 2$ and $\lim x_n = 2$ then $\lim f(x_n) = -3$. Clearly $x_n = 2 + \frac{1}{n}$ satisfies $x_n \neq 2$ and $\lim x_n = 2$, so $\lim f(x_n) = -3$.

(g) If f is continuous at 2 and $f(2 + 1/n) = \arctan(n)$ for all $n \in \mathbb{N}$, then $f(2) = \pi/2$.
Solution. True. By definition of continuity, $\lim_{x\to 2} f(x) = f(2)$. Thus, since $x_n = 2 + \frac{1}{n}$ satisfies $\lim x_n = 2$ we have $\lim f(x_n) = f(2)$. But $\lim f(x_n) = \lim f(2 + 1/n) = \lim \arctan(n) = \pi/2$, so $f(2) = \pi/2$.

4. Suppose a function g is continuous at c and $g(c) = m > 0$. Show that there exists some $\delta > 0$ such that $g(x) < \frac{9}{8}m$ for all $x \in (c - \delta, c + \delta)$.

Solution. Let $\epsilon = \frac{1}{8}m > 0$. Then since g is continuous at c, there exists $\delta > 0$ such that $|g(x) - g(c)| < \frac{1}{8}m$ whenever $|x - c| < \delta$. But $|g(x) - g(c)| < \frac{1}{8}m$ implies $g(x) - g(c) < \frac{1}{8}m$, so $g(x) < g(x) + \frac{1}{8}m = \frac{9}{8}m$.

5. Let f be defined on the interval $[0, 4]$ by $f(x) = \frac{x(4-x)}{2+\cos(x)}$. Prove that there exists some $c \in (0, 4)$ such that $f(c) \geq f(x)$ for all $x \in [0, 4]$.

Solution. Since $2 + \cos(x) \geq 1$ for all x, and both $2 + \cos(x)$ and $x(4 - x)$ are continuous functions, f is continuous on $[0, 4]$. Hence by the Extreme Value Theorem, f attains a maximum value on $[0, 4]$. That is, there is some $c \in [0, 4]$ such that $f(c) \geq f(x)$ for all $x \in [0, 4]$. Since $f(0) = f(4) = 0$ and $f(\pi) = \pi(4 - \pi) > 0$, f does not attain its maximum at either 0 or 4, so $c \in (0, 4)$.

6. Determine which of the following functions are uniformly continuous on the given domain. Prove your assertions.

(a) $f(x) = \sin(1/x)$ on $(0, 2)$.
Solution. f is not uniformly continuous on $(0, 2)$. Let $\epsilon = 1$, and let $\delta > 0$ be given. For each n, define $x_n = \frac{1}{\pi/2+2n\pi}$ and $y_n = \frac{1}{2n\pi}$. Then

$$|x_n - y_n| = \frac{\pi/2}{(\pi/2 + 2n\pi)(2n\pi)} < \delta$$

provided n is large enough, but

$$|f(x_n) - f(y_n)| = |1 - 0| = 1 \geq \epsilon$$

for all n.

(b) $f(x) = \sin(1/x)$ on $[1, 5]$.

Solution. f is continuous on the closed and bounded interval $[1, 5]$, so f is uniformly continuous on $[1, 5]$.

(c) $g(x) = \sin(x)$ on $\mathbb{R}$

Solution. g is uniformly continuous on $\mathbb{R}$. To see this, fist write

$$\begin{aligned}|\sin(y) - \sin(x)| &= |\sin(x + y - x) - \sin(x)| \\ &= |\sin(x)\cos(y - x) + \sin(y - x)\cos(x) - \sin(x)| \\ &= |\sin(x)[\cos(y - x) - 1] + \cos(x)\sin(y - x)| \\ &\le |\cos(y - x) - 1| + |\sin(y - x)|\end{aligned}$$

Now let $\epsilon > 0$ be given. Since $\lim_{t\to 0}\sin(t) = 0$, there exists some $\delta_1 > 0$ such that $|\sin(t)| < \epsilon/2$ whenever $|t| < \delta_1$. Since $\lim_{t\to 0}\cos(t) = 1$, there exists some $\delta_2 > 0$ such that $|\cos(t) - 1| < \epsilon/2$ whenever $|t| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $|y - x| < \delta$ it follows that $|\cos(y - x) - 1| + |\sin(y - x)| < \epsilon/2 + \epsilon/2 = \epsilon$.

(d) $h(x) = \sqrt{x}$ on $[0, \infty)$

Solution. h is uniformly continuous on $[0, \infty)$. The proof is a little tricky since we need to consider a couple of cases. Given $\epsilon > 0$, first suppose either x or y is in $[1, \infty)$. Then $\sqrt{y} + \sqrt{x} \ge 1$, so

$$|\sqrt{y} - \sqrt{x}| = \frac{|y - x|}{\sqrt{y} + \sqrt{x}} \le |y - x|$$

and thus $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \epsilon$. In other words $\delta_1 = \epsilon$ works for all such x and y. On the other hand, if both x and y are in $[0, 1]$, then since f is continuous on $[0, 1]$, it must be uniformly continuous on $[0, 1]$ and thus there exists some $\delta_2 > 0$ such that $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta_2$. Therefore, choosing $\delta = \min\{\delta_1, \delta_2\}$ implies that $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta$.

7. Let

$$f(x) = \begin{cases} ax^2 + bx & x < 1 \\ \frac{1}{x} & x \ge 1 \end{cases}$$

Find a and b so that f is differentiable on $\mathbb{R}$.

Solution. We just need to ensure that f is differentiable at $x = 1$. In order for f to be continuous at $x = 1$ we need $\lim_{x\to 1} f(x) = f(1)$. Since $f(1) = 1$ and

$$\lim_{x\to 1^-} f(x) = a + b$$

we need $a + b = 1$. By the result of Exercise 4.1.11 it then follows that f will be differentiable if $g'(1) = h'(1)$ where $g(x) = ax^2 + bx$ and $h(x) = \frac{1}{x}$. This gives $2a + b = -1$. Solving this system of equations gives $a = -2$ and $b = 3$.

8. Show that the equation $x^2 = 3 + \sin x$ has *exactly* two solutions. (You do not need to find them.)

Solution. See Homework 7 Part B Solutions, #7.

9. Suppose f is continuous on $[a, b]$ and differentiable on (a, b) and that $a \le f(x) \le b$ for all $x \in [a, b]$.

(a) Prove that the equation $f(x) = x$ has at least one solution on $[a, b]$.

Solution. Let $g(x) = f(x) - x$. Then g is continuous on $[a, b]$ and $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. Thus 0 is between $g(a)$ and $g(b)$, so by the Intermediate Value Theorem there exists some $c \in [a, b]$ such that $g(c) = 0$, which implies $f(c) = c$.

(b) Suppose in addition that $f'(x) \neq 1$ for all $x \in (a, b)$. Prove that the equation $f(x) = x$ has exactly one solution on $[a, b]$.

Solution. Suppose there exists two solutions $c_1 \neq c_2$. Then $f(c_1) = c_1$ and $f(c_2) = c_2$. Without loss of generality, suppose $c_1 < c_2$. Then by the Mean Value Theorem, there exists some $c \in (c_1, c_2)$ such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1,$$

a contradiction.

10. Suppose g is differentiable everywhere, $g'(x) \leq 2x + 3$ for all $x > 1$ and $g(1) = 7$. Prove $g(x) \leq x^2 + 3x + 3$ for all $x \geq 1$.

Solution. Let $h(x) = g(x) - x^2 - 3x - 3$. Then h is differentiable everywhere. Given any $x > 1$, the Mean Value Theorem applied to h on the interval $[1, x]$ implies there exists $c \in (1, x)$ such that $h(x) - h(1) = h'(c)(x - 1)$. Since $h'(x) = g'(x) - 2x - 3 \leq 0$ by the assumption about g', we have $h'(c) \leq 0$. Thus $h(x) - h(1) \leq 0$ for all $x \geq 0$. Since $h(1) = 0$, this implies $h(x) \leq 0$ for all $x \geq 0$, and thus $g(x) \leq x^2 + 3x + 3$ for all $x \geq 0$.

11. Suppose f is differentiable everywhere, and there exists $M > 0$ such that $|f'(x)| \leq M$ for all x. Prove f is uniformly continuous on $\mathbb{R}$.

Solution. Given $\epsilon > 0$, choose $\delta = \epsilon/M$. For any $x, y \in \mathbb{R}$, the Mean Value Theorem implies there exists c between x and y such that $f(y) - f(x) = f'(c)(y - x)$. Thus, since $|f'(c)| \leq M$, we have

$$|f(y) - f(x)| = |f'(c)||y - x| \leq M|y - x| < M\delta = \epsilon$$

for all x, y such that $|y - x| < \delta$.