College of the Holy Cross, Fall Semester, 2016
Math 242, Midterm 3 Practice Questions

1. (a) Write the definition of $\lim_{x \to c} f(x) = L$.
   (b) Use the definition to prove that $\lim_{x \to 3} x^2 + 2x = 15$.
   (c) Use the definition to prove that $\lim_{x \to 8} x^{1/3} = 2$.

2. Let $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 2x^2 & x \in \mathbb{Q}^c \end{cases}$
   (a) Prove that $\lim_{x \to 0} f(x) = 0$.
   (b) Use appropriately chosen sequences to prove that $\lim_{x \to 1} f(x)$ does not exist.

3. True/False. Prove your assertions.
   (a) If f is bounded and continuous on $(0, 1)$, then f attains a maximum value on $(0, 1)$.
   (b) If $\lim_{x \to 5} f(x) = 0.3$, then there exists some $\delta > 0$ such that $f(x) > 0.28$ whenever $0 < |x - 5| < \delta$.
   (c) If $f(0) = 2$ and $f(3) = 5$, then there exists some $c \in (0, 3)$ such that $f(c) = 3$.
   (d) The function $f(x) = \frac{\sin(x^2+1)}{2+\cos(3x)}$ attains a maximum value on $[-50, 50]$.
   (e) If $\lim f(2 + 1/n) = -3$, then $\lim_{x \to 2} f(x) = -3$.
   (f) If $\lim_{x \to 2} f(x) = -3$, then $\lim f(2 + 1/n) = -3$.
   (g) If f is continuous at 2 and $f(2+1/n) = \arctan(n)$ for all $n \in \mathbb{N}$, then $f(2) = \pi/2$.

4. Suppose a function g is continuous at c and $g(c) = m > 0$. Show that there exists some $\delta > 0$ such that $g(x) < \frac{9}{8}m$ for all $x \in (c - \delta, c + \delta)$.

5. Let f be defined on the interval $[0, 4]$ by $f(x) = \frac{x(4-x)}{2+\cos(x)}$. Prove that there exists some $c \in (0, 4)$ such that $f(c) \geq f(x)$ for all $x \in [0, 4]$.

6. Determine which of the following functions are uniformly continuous on the given domain. Prove your assertions.
   (a) $f(x) = \sin(1/x)$ on $(0, 2)$.
   (b) $f(x) = \sin(1/x)$ on $[1, 5]$.
   (c) $g(x) = \sin(x)$ on $\mathbb{R}$
   (d) $h(x) = \sqrt{x}$ on $[0, \infty)$

7. Let

$$f(x) = \begin{cases} ax^2 + bx & x < 1 \\ \frac{1}{x} & x \geq 1 \end{cases}$$

Find a and b so that f is differentiable on $\mathbb{R}$.

8. Show that the equation $x^2 = 3 + \sin x$ has *exactly* two solutions. (You do not need to find them.)

9. Suppose f is continuous on $[a, b]$ and differentiable on (a, b) and that $a \leq f(x) \leq b$ for all $x \in [a, b]$.

   (a) Prove that the equation $f(x) = x$ has at least one solution on $[a, b]$.

   (b) Suppose in addition that $f'(x) \neq 1$ for all $x \in (a, b)$. Prove that the equation $f(x) = x$ has exactly one solution on $[a, b]$.

10. Suppose g is differentiable everywhere, $g'(x) \leq 2x+3$ for all $x > 1$ and $g(1) = 7$. Prove $g(x) \leq x^2 + 3x + 3$ for all $x \geq 1$.

11. Suppose f is differentiable everywhere, and there exists $M > 0$ such that $|f'(x)| \leq M$ for all x. Prove f is uniformly continuous on $\mathbb{R}$.