College of the Holy Cross, Fall Semester, 2016
Math 242, Midterm 2 Practice Questions
Solutions

1. Give an example of each of the following, or explain why such an example is not possible.

   (a) An unbounded Cauchy sequence.

   Solution. Impossible. Cauchy sequences converge, and convergent sequences are bounded.

   (b) A bounded sequence that has no convergent subsequences.

   Solution. Impossible. The Bolzano-Weierstrass Theorem states that any bounded sequence has a convergent subsequence.

   (c) A convergent sequence that has a subsequence that converges to 5 and a subsequence that converges to 7.

   Solution. Impossible. Every subsequence of a convergent sequence converges to the limit of the sequence.

   (d) Series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ that both diverge, such that the series $\sum_{n=1}^{\infty} a_n + b_n$ converges.

   Solution. Let $a_n = 1$ and $b_n = -1$. Then $\sum_{n=1}^{\infty} 1$ and $\sum_{n=1}^{\infty}(-1)$ both diverge, but $\sum_{n=1}^{\infty}(1 + -1) = \sum_{n=1}^{\infty} 0$ converges.

   (e) A series $\sum_{n=1}^{\infty} a_n$ that converges and a series $\sum_{n=1}^{\infty} b_n$ that diverges, such that the series $\sum_{n=1}^{\infty} a_n + b_n$ converges.

   Solution. This is not possible. For if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n + b_n$ both converge, since $b_n = (a_n + b_n) - a_n$ it follows that $\sum_{n=1}^{\infty} b_n$ also converges.

   (f) An unbounded sequence x_n with $x_n \geq 0$ for all n such that $\lim x_n \neq +\infty$.

   Solution. Let $x_n = 0, 1, 0, 2, 0, 3, 0, 4, \ldots$. Then x_n is unbounded and $x_n \geq 0$ for all n, but x_n does not diverge to $+\infty$.

2. Use the definition of an infinite limit to prove that $\lim \sqrt{1 + \sqrt{n}} = +\infty$.

   Solution. Given $M > 0$ we want $\sqrt{1 + \sqrt{n}} > M$, which is equivalent to $1 + \sqrt{n} > M^2$, which is in turn equivalent to $n > (M^2 - 1)^2$. By the Archimedian Property, there exists some $n_0 \in \mathbb{N}$ such that $n_0 > (M^2 - 1)^2$. Then for any $n \geq n_0$ we have $n > (M^2 - 1)^2$ and thus $\sqrt{1 + \sqrt{n}} > M$.

3. Suppose $\lim x_n = +\infty$ and $\lim \frac{x_n}{y_n} = 3$. Prove that $\lim y_n = +\infty$.

   Solution. Since $\frac{x_n}{y_n} \to 3$, if we choose $\epsilon = 1$, then there is some $n_0 \in \mathbb{N}$ such that $\left|\frac{x_n}{y_n} - 3\right| < 1$ whenever $n \geq n_0$. But $\left|\frac{x_n}{y_n} - 3\right| < 1$ implies $2 < \frac{x_n}{y_n} < 4$, and thus $y_n > \frac{1}{4}x_n$. Now since $x_n \to +\infty$, for any $M > 0$ there is some $n_1 \in \mathbb{N}$ such that $x_n > 4M$ whenever $n \geq n_1$. Let $n_2 = \max\{n_0, n_1\}$. Then whenever $n \geq n_2$ we have $y_n > \frac{1}{4}x_n$ and $x_n > 4M$ so $y_n > \frac{1}{4}(4M) = M$. Hence $y_n \to +\infty$.

4. Find the sum of each series. Explain your reasoning.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{9^{n-1} + 10^{n+1}}{11^n}$

Solution. Rewrite this as the sum of two geometric series:

$$\sum_{n=0}^{\infty} \frac{9^{n-1}}{11^n} + \sum_{n=0}^{\infty} \frac{10^{n+1}}{11^n} = \sum_{n=0}^{\infty} \frac{1}{9}(9/11)^n + \sum_{n=0}^{\infty} 10(10/11)^n = \frac{1/9}{1-9/11} + \frac{10}{1-10/11} = \frac{11}{18} + 110$$

(b) $\displaystyle\sum_{n=1}^{\infty} \cos(\pi/n) - \cos(\pi/(n+2))$

Solution. This series is telescoping. To see this, write the n^{th} partial sum:

$$\begin{aligned} s_n &= \cos(\pi) - \cos(\pi/3) + \cos(\pi/2) - \cos(\pi/4) \\ &\quad + \cos(\pi/3) - \cos(\pi/5) + \cos(\pi/4) - \cos(\pi/6) + \cdots \\ &\quad + \cos(\pi/(n-1)) - \cos(\pi/(n+1)) + \cos(\pi/n) - \cos(\pi/(n+2)) \end{aligned}$$

All the terms in the partial sum cancel, except the $\cos(\pi)$, $\cos(\pi/2)$, $\cos(\pi/(n+1))$ and $\cos(\pi/(n+2)$. Thus

$$s_n = \cos(\pi) + \cos(\pi/2) - \cos(\pi/(n+1)) - \cos(\pi/(n+1))$$

so the sum of the series is

$$s = \lim s_n = \cos(\pi) + \cos(\pi/2) - \cos(0) - \cos(0) = -1 + 0 - 1 - 1 = -3.$$

5. Suppose a_n and b_n are nonnegative for all n, $\lim \frac{a_n}{b_n} = \frac{3}{4}$ and $\displaystyle\sum_{n=1}^{\infty} b_n = 8$. Which of the following statements are true?

   (i) $\displaystyle\sum_{n=1}^{\infty} a_n = 6.$

   (ii) $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

   (iii) $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

   (iv) This doesn't tell us anything about $\displaystyle\sum_{n=1}^{\infty} a_n$.

   Solution. (ii) By the limit comparison test, since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and $\lim \frac{a_n}{b_n} > 0$, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

6. State whether each of the following series converges or diverges. Prove your assertion.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{4+n}{1+n^3}$

   Solution. For large n, the terms of this series are roughly $\frac{1}{n^2}$. Since

$$\lim \frac{\frac{4+n}{1+n^3}}{\frac{1}{n^2}} = \lim \frac{\frac{4}{n}+1}{\frac{1}{n^3}+1} = 1 \neq 0$$

   and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the Limit Comparison Test implies that the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution. This is an alternating series, so let's apply the Alternating Series Test. Let $a_n = \frac{n}{n^2+1}$. Then we have $a_{n+1} < a_n$ provided

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}.$$

For $n \geq 1$, this inequality holds if and only if $(n+1)(n^2+1) < n((n+1)^2+1)$. Expanding both sides of this, we have $n^3+n^2+n+1 < n^3+2n^2+n+1$, which is equivalent to $n^2 > 0$. This is clearly true, so all of these inequalities are true for $n \geq 1$, and thus the sequence a_n is decreasing. Since

$$\lim a_n = \lim \frac{n}{n^2+1} = \lim \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0$$

the series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n+3}$

Solution. Let's use the limit comparison test with $\frac{1}{n}$.

$$\lim \frac{\frac{2}{5n+3}}{\frac{1}{n}} = \lim \frac{2}{\frac{5}{n}+3} = \frac{2}{3} \neq 0.$$

Thus since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right)$

Solution. $\lim \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1 \neq 0$ so the series diverges by the n^{th} term test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Solution. Apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{4n+2}$$

Therefore

$$r = \lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4} < 1$$

so the series converges.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$

Solution. The terms of this series are positive and decreasing, so we may apply the Cauchy Condensation Test. Setting $a_n = \frac{(\ln n)^2}{n^2}$, we have

$$b_n = 2^n a_{2^n} = \frac{2^n(\ln(2^n))^2}{(2^n)^2} = \frac{(n \ln 2)^2}{2^n} = \frac{(\ln 2)^2 n^2}{2^n}.$$

Since
$$\lim\left|\frac{b_{n+1}}{b_n}\right| = \lim\frac{(n+1)^2}{2n^2} = \frac{1}{2} < 1,$$
the Ratio Test implies that the series $\sum_{n=0}^{\infty} b_n$ converges, and thus the series $\sum_{n=1}^{\infty} a_n$ converges.

7. Suppose $a_n > 0$ for all n and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges.

Solution. Since $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, the n^{th} term test implies that $\lim a_n = 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $a_n < 1$ for all $n \geq n_0$. Since $a_n > 0$ as well, this implies $a_n^2 < a_n$ for all $n \geq n_0$. Hence the comparison test implies $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges.

8. Find the interval of convergence of each power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$

Solution. Using the ratio test,
$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{\sqrt{n}|x+2|}{\sqrt{n+1}} = |x+2|$$
and thus the series converges if $|x+2| < 1$ and diverges if $|x+2| > 1$. Next, $|x+2| = 1$ if $x = -1$ or $x = -3$. When $x = -1$ the series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and when $x = -3$ it is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. The first series diverges since it is a p-series with $p = \frac{1}{2} < 1$ and the second series converges by the alternating series test. thus the interval of convergence is $[-3, -1)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n + 3^n}$

Solution. Using the ratio test again, and the fact that $\lim(2/3)^n = 0$ we have
$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{|x|(2^n+3^n)}{2^{n+1}+3^{n+1}} = \lim\frac{|x|((2/3)^n+1)}{2\cdot(2/3)^n+3} = \frac{|x|}{3},$$
and thus the series converges if $|x| < 3$ and diverges if $|x| > 3$. When $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$ and when $x = -3$ is becomes $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^n}{2^n+3^n}$. Since
$$\lim\frac{3^n}{2^n+3^n} = \lim\frac{1}{(2/3)^n+1} = 1 \neq 0,$$
both of these series diverge by the n^{th} term test. Thus the interval of convergence is $(-3, 3)$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$

Solution. Applying the ratio test once more gives
$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{|x|(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}|x|,$$

so the series converges when $|x| < 4$ and diverges when $|x| > 4$. When $x = 4$ the series becomes $\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$. By the above calculation,

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)^2}{(2n+1)(2n+2)} = \frac{2n+2)}{2n+1} > 1$$

for all n. Hence $a_{n+1} > a_n$ for all n, so the sequence a_n is strictly increasing. Since $a_1 = 2$ this implies $a_n > 2$ for all n, so $\lim a_n \neq 0$ and thus the series diverges by the n^{th} term test. The same holds when $x = -4$, so the interval of convergence is $(-4, 4)$.

9. Suppose that $\lim_{n\to\infty} a_n = \dfrac{3}{5}$. Prove that the interval of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $(-1, 1)$.

Solution. Let $b_n = a_n x^n$. Then

$$\lim \left|\frac{b_{n+1}}{b_n}\right| = \lim \frac{a_{n+1}|x|}{a_n} = \lim \frac{\frac{3}{5}|x|}{\frac{3}{5}} = |x|,$$

so the ratio test implies that the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = 1$, the series becomes $\sum_{n=0}^{\infty} a_n$, which diverges by the n^{th} term test since $\lim a_n = \frac{3}{5} \neq 0$. Likewise, when $x = -1$ the series becomes $\sum_{n=0}^{\infty} a_n(-1)^n$, which also diverges since $\lim a_n(-1)^n$ does not exist. Thus the series converges if and only if $x \in (-1, 1)$.