

College of the Holy Cross, Fall 2016
Math 242, Midterm 1 Practice Questions
Solutions

1. Use Axioms 1 through 9 to prove that if $x \cdot y = x \cdot z$ and $x \neq 0$ then $y = z$.

Solution. Suppose $x \cdot y = x \cdot z$. Since $x \neq 0$, Axiom 5(b) implies that there exists a real number x^{-1} such that $x \cdot x^{-1} = 1$. By Axiom 1, this implies $x^{-1} \cdot x = 1$, so we have

$$\begin{array}{ll}
 y = y \cdot 1 & \text{by Axiom 4(b)} \\
 = 1 \cdot y & \text{by Axiom 1} \\
 = (x^{-1} \cdot x) \cdot y & \text{since } x^{-1} \cdot x = 1 \\
 = x^{-1} \cdot (x \cdot y) & \text{by Axiom 2} \\
 = x^{-1} \cdot (x \cdot z) & \text{by hypothesis} \\
 = (x^{-1} \cdot x) \cdot z & \text{by Axiom 2} \\
 = 1 \cdot z & \text{since } x^{-1} \cdot x = 1 \\
 = z \cdot 1 & \text{by Axiom 1} \\
 = z & \text{by Axiom 4(b).}
 \end{array}$$

2. (a) Show that $\sqrt{3}$ is irrational.

Solution. Suppose $\sqrt{3}$ is rational. Then we could write $\sqrt{3} = \frac{m}{n}$, where m and n are integers that have no common factors. This implies $m^2 = 3n^2$, so m^2 is a multiple of 3. Now there are 3 possibilities for m : $m = 3k$, $m = 3k + 1$ or $m = 3k + 2$ for some integer k . If $m = 3k + 1$, then $m^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. But this contradicts the fact that m^2 is a multiple of 3. Likewise, if $m = 3k + 2$, then $m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, again contradicting the fact that m^2 is a multiple of 3. The only possibility therefore is that $m = 3k$ for some integer k . This then implies $9k^2 = 3n^2$, so $n^2 = 3k^2$. Hence n^2 is a multiple of 3, and by reasoning as before, it follows that n is a multiple of 3. But this means that both m and n share the common factor of 3, contrary to the assumption. This contradiction implies that $\sqrt{3}$ is irrational.

- (b) Suppose $t > 0$ is irrational. Prove that $\sqrt{t}$ is irrational.

Solution. Suppose instead that $\sqrt{t}$ is rational. Then $\sqrt{t} = p/q$ where p and q are integers and $q \neq 0$. Then $t = p^2/q^2$ would also be rational, a contradiction.

3. Fix $r \neq 1$. Use the principle of induction to prove that the summation formula

$$\sum_{k=0}^n r^k = \frac{1 - r^{n+1}}{1 - r}$$

holds for all $n \in \mathbb{N}$.

Solution. Denote this summation formula by S_n . When $n = 1$, the statement reads $1 + r = \frac{1-r^2}{1-r}$, which is true since $1 - r^2 = (1 - r)(1 + r)$. Next suppose S_n is true for some $n \geq 1$. Then

$$\begin{aligned} \sum_{k=0}^{n+1} r^k &= \left(\sum_{k=0}^n r^k \right) + r^{n+1} \\ &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} && \text{since } S_n \text{ is true} \\ &= \frac{1 - r^{n+1} + r^{n+1}(1 - r)}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r}, \end{aligned}$$

so S_{n+1} is true. Thus S_n implies S_{n+1} and therefore by the principle of induction S_n is true for all $n \geq 1$.

4. Let $A = \{3 - \frac{1}{n} : n \in \mathbb{N}\}$. Find $\text{lub } A$, and prove your assertion.

Solution. $\text{lub } A = 3$. First, since $3 - \frac{1}{n} < 3$ for any n , 3 is an upper bound for A . Now consider any number $M < 3$. Since $3 - M > 0$, the Archimedian Property implies that there is some $n \in \mathbb{N}$ such that $\frac{1}{n} < 3 - M$. This implies $3 - \frac{1}{n} > M$, so M is not an upper bound of A . Therefore every upper bound M of A must satisfy $M \geq 3$. Hence 3 is the least upper bound of A .

5. Let $A = \{x \in \mathbb{R} \mid x^5 - 2x < 1000\}$.

(a) Prove that A is bounded above.

Solution. 10 is an upper bound for A . To see this, suppose 10 is not an upper bound for A . Then for some $x \in A$ we have $x > 10$. But this would imply $x^4 > 10^4 = 10000$, so $x^4 - 2 > 9998$ and thus $x^5 - 2x = x(x^4 - 2) > 10(9998) > 1000$. This contradicts the fact that $x^5 - 2x < 1000$ for each $x \in A$. Hence 10 is an upper bound for A .

(b) Prove that A has a least upper bound.

Solution. By the previous problem A is bounded above by 10. Since $0^5 - 2(0) = 0 < 1000$, $0 \in A$, and thus A is nonempty. The Least Upper Bound Axiom then implies that A has a least upper bound.

6. Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded above and satisfy $\text{lub } A < \text{lub } B$. Prove that there exists some $y \in B$ such that $x < y$ for every $x \in A$.

Solution. Since $\text{lub}(A) < \text{lub}(B)$, $\text{lub}(A)$ is not an upper bound for B . Hence there exists some $y \in B$ such that $\text{lub}(A) < y$. Since $\text{lub}(A)$ is an upper bound for A , $x \leq \text{lub}(A)$ for all $x \in A$. By transitivity it follows that $x < y$ for all $x \in A$.

7. (a) Complete the following definition. A sequence x_n converges to a real number a if

Solution. For any $\epsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ for all $n \geq n_0$.

- (b) Use the definition of convergence to prove that $\lim_{n \rightarrow \infty} \frac{3n}{2n-1} = \frac{3}{2}$.

Solution. Let $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{3}{2\epsilon}$ (this is possible by the Archimedian property since $\frac{3}{2\epsilon} > 0$). Then for any $n \geq n_0$, we have $n > \frac{3}{2\epsilon}$, so $\frac{3}{2n} < \epsilon$ and thus

$$\begin{aligned} \left| \frac{3n}{2n-1} - \frac{3}{2} \right| &= \frac{3}{2(2n-1)} \\ &\leq \frac{3}{2(n)} && \text{since } 2n-1 \geq n \text{ when } n \geq 1 \\ &< \epsilon \end{aligned}$$

for all $n \geq n_0$.

8. Suppose $\lim a_n = 7$. Show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $a_n > 6.99$.

Solution. Let $\epsilon = 0.01$. Then since $\lim a_n = 7$, there exists some n_0 such that $|a_n - 7| < 0.01$ for all $n \geq n_0$. This implies $-0.01 < a_n - 7 < 0.01$, or equivalently $6.99 < a_n < 7.01$. Hence $a_n > 6.99$ for all $n \geq n_0$.

9. True or False. If True, give a short proof. If False, give a counterexample.

- (a) If the sequences $\{x_n\}$ and $\{y_n\}$ both diverge, then the sequence $\{x_n y_n\}$ diverges.

Solution. False, $x_n = y_n = (-1)^n$ both diverge, but $x_n y_n = 1$ converges.

- (b) If $r \neq 0$ is rational, and t is irrational, then t/r is irrational.

Solution. True. Suppose t/r is rational. Then $t/r = m/n$ for some integers m and n with $n \neq 0$. Solving for t then gives $t = rm/n$. Since r is rational, $r = p/q$ where p and q are integers and $q \neq 0$. Thus $t = (mp)/(nq)$. Since mp and nq are integers and $nq \neq 0$ it follows that t is rational, a contradiction. Hence t/r is irrational.

10. Suppose x_n converges to 0. Prove that $\sqrt[3]{x_n}$ converges to 0.

Solution. Let $\epsilon > 0$. Since $x_n \rightarrow 0$, there exists some $n_0 \in \mathbb{N}$ such that $|x_n| < \epsilon^3$ for all $n \geq n_0$. Thus $|\sqrt[3]{x_n}| < \epsilon$ for all $n \geq n_0$.

11. Suppose $\lim x_n = 0$ and y_n is bounded. Prove that $\lim x_n y_n = 0$.

Solution. Since y_n is bounded, there exists a real number $M > 0$ such that $|y_n| \leq M$ for all n . Let $\epsilon > 0$. Since $x_n \rightarrow 0$, there exists some $n_0 \in \mathbb{N}$ such that $|x_n| < \frac{\epsilon}{M}$ for all $n \geq n_0$. Thus

$$|x_n y_n - 0| = |x_n| |y_n| \leq M |x_n| < M \cdot \frac{\epsilon}{M} = \epsilon$$

for all $n \geq n_0$. Hence $\lim x_n y_n = 0$.

12. Let x_n be a sequence with the property that $x_n^2 - 5x_n$ converges to 14.

- (a) If x_n converges what are the only possible values of its limit?

Solution. Let $a = \lim x_n$. By the Algebraic Limit Theorem, $\lim x_n^2 - 5x_n = a^2 - 5a$, so $a^2 - 5a = 14$. The solutions of this equation are $a = 7$ and $a = -2$.

- (b) Must x_n converge?

Solution. No. The sequence $x_n = 7, -2, 7, -2, 7, -2, \dots$ satisfies $x_n^2 - 5x_n = 14$ for all n , so $x_n^2 - 5x_n$ converges to 14, but x_n does not converge.

13. Consider the sequence defined recursively by $x_1 = 1$ and $x_{n+1} = \frac{x_n^2 + 8}{6}$.

- (a) Prove that the sequence x_n is increasing.

Solution. By induction. Let S_n denote the statement $x_n \leq x_{n+1}$. Since $x_1 = 1 < \frac{9}{6} = x_2$, this proves S_1 is true. Now suppose $x_k \leq x_{k+1}$ for some $k \geq 1$. Then since both x_k and x_{k+1} are positive, this implies $x_k^2 \leq x_{k+1}^2$. Adding 8 and dividing by 6 then gives $\frac{x_k^2 + 8}{6} \leq \frac{x_{k+1}^2 + 8}{6}$, which implies $x_{k+1} \leq x_{k+2}$. Thus $S_k \implies S_{k+1}$ for all $k \geq 1$, so by the principle of induction, S_n is true for all $n \in \mathbb{N}$. Hence x_n is increasing.

- (b) Prove that $x_n \leq 3$ for all n .

Solution. By induction again. Let S_n denote the statement $x_n \leq 3$. The base case S_1 is true since $x_1 = 1 < 3$. So suppose $x_k \leq 3$ for some $k \geq 1$. Then since x_k is positive, $x_k^2 \leq 9$. Adding 8 and dividing by 6 then gives $\frac{x_k^2 + 8}{6} \leq \frac{17}{6} < 3$. Thus $x_{k+1} \leq 3$. Hence $S_k \implies S_{k+1}$ for all $k \geq 1$, so by the principle of induction S_n is true for all $n \in \mathbb{N}$.

- (c) Prove that the sequence x_n converges, and find its limit.

Solution. By parts (a) and (b), the sequence x_n is monotone and bounded (above by 3 and below by $x_1 = 1$ since x_n is increasing), so x_n converges by the Monotone Convergence Theorem. Let $a = \lim x_n$. Then, using the algebraic limit theorem,

$$a = \lim x_{n+1} = \lim \frac{x_n^2 + 8}{6} = \frac{a^2 + 8}{6},$$

so $a^2 + 8 = 6a$ which implies $a = 2$ or $a = 4$. But since $x_n \leq 3$ for all n , we have $a \leq 3$. Thus $a = 2$.

14. Consider the sequence defined recursively by $y_1 = 5$ and $y_{n+1} = \frac{y_n^2 + 8}{6}$.

- (a) Use induction to prove that $y_n \geq 5$ for all n .

Solution. When $n = 1$, we have $y_1 = 5 \geq 5$. Now suppose $y_n \geq 5$ for some $n \geq 1$. Then $y_n^2 \geq 25$, so $y_n^2 + 8 \geq 33$ and therefore $y_{n+1} = \frac{y_n^2 + 8}{6} \geq \frac{33}{6} > 5$, so $y_{n+1} \geq 5$.

- (b) Prove by contradiction that y_n diverges.

Solution. Suppose y_n converges. Let $a = \lim y_n$. Then by the Algebraic Limit Theorem, $\lim y_{n+1} = \lim \frac{y_n^2 + 8}{6} = \frac{a^2 + 8}{6}$. But $\lim y_{n+1} = \lim y_n = a$, so we have $a = \frac{a^2 + 8}{6}$. The solutions of this equation are $a = 2$ and $a = 4$. But since $y_n \geq 5$ for all n , a must be at least 5, a contradiction.

15. Determine whether or not each sequence converges, and find the limit of those that converge.

(a) $x_n = \frac{3}{5n^2+4}$ if n is even and $x_n = \frac{n}{1-5n}$ if n is odd.

Solution. Using the fact that $\lim \frac{1}{k} = \lim \frac{1}{k^2} = 0$ together with the algebraic limit theorem, we have

$$\lim x_{2k} = \lim \frac{3}{5(2k)^2 + 4} = \lim \frac{\frac{3}{k^2}}{20 + \frac{4}{k^2}} = \frac{0}{20} = 0$$

and

$$\lim x_{2k+1} = \lim \frac{2k+1}{1-5(2k+1)} = \lim \frac{2 + \frac{1}{k}}{-\frac{4}{k} - 10} = \frac{2}{-10} = -\frac{1}{5}.$$

Thus x_n has subsequences that converge to different limits, so x_n does not converge.

(b) $x_n = \frac{3n}{5n+4}$ if n is even and $x_n = \frac{1-3n}{1-5n}$ if n is odd.

Solution. Using the fact that $\lim \frac{1}{k} = \lim \frac{1}{k^2} = 0$ together with the algebraic limit theorem, we have

$$\lim x_{2k} = \lim \frac{6k}{10k+4} = \lim \frac{6}{10 + \frac{4}{k}} = \frac{6}{10} = \frac{3}{5}$$

and

$$\lim x_{2k+1} = \lim \frac{1-3(2k+1)}{1-5(2k+1)} = \lim \frac{-6 - \frac{2}{k}}{-\frac{5}{k} - 10} = \frac{-6}{-10} = \frac{3}{5}.$$

Therefore, by the theorem we proved in class, since both x_{2k} and x_{2k+1} converge to $\frac{3}{5}$, x_n converges to $\frac{3}{5}$.