College of the Holy Cross, Fall 2016
Math 242, Midterm 1 Practice Questions

1. Use Axioms 1 though 9 to prove that if $x \cdot y = x \cdot z$ and $x \neq 0$ then $y = z$.

2. (a) Show that $\sqrt{3}$ is irrational.
   (b) Suppose $t > 0$ is irrational. Prove that $\sqrt{t}$ is irrational.

3. Fix $r \neq 1$. Use the principle of induction to prove that the summation formula

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

   holds for all $n \in \mathbb{N}$.

4. Let $A = \{3 - \frac{1}{n} : n \in \mathbb{N}\}$. Find $\operatorname{lub} A$, and prove your assertion.

5. Let $A = \{x \in \mathbb{R} \mid x^5 - 2x < 1000\}$.
   (a) Prove that A is bounded above.
   (b) Prove that A has a least upper bound.

6. Suppose A and B are nonempty subsets of $\mathbb{R}$ that are bounded above and satisfy $\operatorname{lub} A < \operatorname{lub} B$. Prove that there exists some $y \in B$ such that $x < y$ for every $x \in A$.

7. (a) Complete the following definition. A sequence x_n to converges to a real number a if
   (b) Use the definition of convergence to prove that $\displaystyle\lim_{n\to\infty} \frac{3n}{2n-1} = \frac{3}{2}$.

8. Suppose $\lim a_n = 7$. Show that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $a_n > 6.99$.

9. True or False. If True, give a short proof. If False, give a counterexample.
   (a) If the sequences $\{x_n\}$ and $\{y_n\}$ both diverge, then the sequence $\{x_n y_n\}$ diverges.
   (b) If $r \neq 0$ is rational, and t is irrational, then t/r is irrational.

10. Suppose x_n converges to 0. Prove that $\sqrt[3]{x_n}$ converges to 0.

11. Suppose $\lim x_n = 0$ and y_n is bounded. Prove that $\lim x_n y_n = 0$.

12. Let x_n be a sequence with the property that $x_n^2 - 5x_n$ converges to 14.
    (a) If x_n converges what are the only possible values of its limit?
    (b) Must x_n converge?

13. Consider the sequence defined recursively by $x_1 = 1$ and $x_{n+1} = \frac{x_n^2 + 8}{6}$.

(a) Prove that the sequence x_n is increasing.

(b) Prove that $x_n \leq 3$ for all n.

(c) Prove that the sequence x_n converges, and find its limit.

14. Consider the sequence defined recursively by $y_1 = 5$ and $y_{n+1} = \dfrac{y_n^2 + 8}{6}$.

    (a) Use induction to prove that $y_n \geq 5$ for all n.

    (b) Prove by contradiction that y_n diverges.

15. Determine whether or not each sequence converges, and find the limit of those that converge.

    (a) $x_n = \frac{3}{5n^2+4}$ if n is even and $x_n = \frac{n}{1-5n}$ if n is odd.

    (b) $x_n = \frac{3n}{5n+4}$ if n is even and $x_n = \frac{1-3n}{1-5n}$ if n is odd.

16. I will ask you to write a complete proof of **one** of the following.

    (a) For every $a > 0$, there exists a real number $b > 0$ such that $b^2 = a$.

    (b) If $\lim x_n = a$ and $\lim y_n = b$, then $\lim x_n + y_n = a + b$

    (c) If x_n is bounded and monotone, then x_n converges.