**College of the Holy Cross, Fall 2016**
**Math 242, Principles of Analysis**
**Cauchy Condensation Test**

The following test for convergence is known as the **Cauchy Condensation Test**. It applies to series with nonnegative, decreasing terms.

**Theorem.** Suppose $a_n \geq 0$ for all $n$ and $a_n$ is a decreasing sequence. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

**Note.** This is an if an only if statement, so this means that either both series converge or both series diverge. Thus the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ is determined by what happens with the series $\sum_{n=0}^{\infty} 2^n a_{2^n}$.

**Proof.** Let

$$s_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad t_n = \sum_{k=0}^{n} 2^k a_{2^k}$$

be the partial sums of the two series. First suppose the series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. Then the sequence $t_n$ converges and is therefore bounded, so there is some $M$ such that $t_n \leq M$ for all $n$. Now observe that, since the sequence $a_n$ is decreasing,

$$\begin{aligned}
s_1 &= a_1 = t_0 \\
s_3 &= a_1 + (a_2 + a_3) \leq a_1 + 2a_2 = t_1 \\
s_7 &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \leq a_1 + 2a_2 + 4a_4 = t_2 \\
s_{15} &\leq a_1 + 2a_2 + 4a_4 + 8a_8 = t_3
\end{aligned}$$

and in general $s_{2^n-1} \leq t_{n-1} \leq M$. This proves that the subsequence $s_{2^n-1}$ is bounded. But since $s_n$ is increasing, this implies that $s_n$ is bounded. So by the Monotone Convergence Theorem, $s_n$ converges.

Conversely, suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then $s_n$ converges and is therefore bounded, so there exists $M$ such that $s_n \leq M$ for all $n$. Now notice that

$$\begin{aligned}
2s_2 &= 2a_1 + 2a_2 = a_1 + (a_1 + 2a_2) = a_1 + t_1 \\
2s_4 &= a_1 + (a_1 + 2a_2 + 2a_3 + 2a_4) \geq a_1 + (a_1 + 2a_2 + 4a_4) = a_1 + t_2 \\
2s_8 &= a_1 + (a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8) \geq a_1 + (a_1 + 2a_2 + 4a_4 + 8a_8) = a_1 + t_3 \\
2s_{16} &\geq a_1 + (a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16}) = a_1 + t_4
\end{aligned}$$

and in general $2s_{2^n} \geq a_1 + t_n$. Thus $t_n \leq 2s_{2^n} \leq 2M$ for all $n$, so the sequence $t_n$ is bounded. Since $t_n$ is increasing, the Monotone Convergence Theorem implies that $t_n$ converges.

**Example.** Let's apply the Cauchy Condensation Test to the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for some $p > 0$.

Here $a_n = \frac{1}{n^p}$, so

$$2^n a_{2^n} = \frac{2^n}{(2^n)^p} = (2^n)^{1-p} = (2^{1-p})^n.$$

Thus

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} (2^{1-p})^n.$$

This is a geometric series with ratio $r = 2^{1-p}$. It therefore converges when $2^{1-p} < 1$ and diverges when $2^{1-p} \geq 1$. Thus the $p$-series converges when $p > 1$ and diverges when $p \leq 1$.