Group Discussion # 3:
Changing Bases – How does this affect the Matrix for a Linear Transformation?

Solutions due: Wednesday, April 7th, at the *beginning* of class
One write-up per group please, **following the same guidelines as the homework.**

Background:

- On Wednesday, we worked with the following problem: Given a vector space V with two different choices of basis, α and α', and given a vector $\vec{v} \in V$,

  how does $[\vec{v}]_{\alpha'}$, the coordinate vector of $\vec{v}$ with respect to the basis α', relate to $[\vec{v}]_{\alpha}$, the coordinate vector of $\vec{v}$ with respect to the basis α?

  Our solution was the following:

  Proposition 1. With the hypotheses as above,

  $$[\vec{v}]_{\alpha'} = [I]_{\alpha}^{\alpha'} [\vec{v}]_{\alpha},$$

  where $I : V \to V$ is the identity transformation.

- We refer to $[I]_{\alpha}^{\alpha'}$ as a *change of basis matrix.* When the *second* basis $\alpha' = \varepsilon$, the standard basis, we saw that the entries of $[I]_{\alpha}^{\varepsilon}$ are just the coordinates of the basis vectors from α, put in as the columns. That is, we have the following:

  Proposition 2. With the hypotheses as above,

  $$[I]_{\alpha}^{\varepsilon} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

  where the entries in the jth column are the coordinates of the jth basis vector $\vec{v_j}$ in α, with respect to ε; that is, where $\vec{v_j} = (a_{1j}, a_{2j}, \ldots, a_{nj})$. for $j = 1, \ldots, n$.

- Today, your job is to work out the related problem: Given a vector space V with two different choices of basis, α and α', and given a linear transformation $T : V \to V$,

  how does $[T]_{\alpha'}^{\alpha'}$, the matrix of T with respect to the basis α', relate to $[T]_{\alpha}^{\alpha}$, the matrix of T with respect to the basis α?

Throughout, we will use the following set-up:

- $T : V \to V$ is a linear transformation,
- V is a finite-dimensional vector space, and
- $\alpha = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ and $\alpha' = \{\vec{v_1'}, \vec{v_2'}, \ldots, \vec{v_n'}\}$ are two different bases for V.
- $\varepsilon = \{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$ is the standard basis, in the case where $V = \mathbb{R}^n$.

We will also need the following proposition regarding the *composition of two linear transformations*:

Proposition 3. Let $S : V \to V$ and $T : V \to V$ be linear transformations, where V is a finite-dimensional vector space. Let δ, β, and γ be any choices of bases for V. Then

$$[TS]_\delta^\gamma = [T]_\beta^\gamma \, [S]_\delta^\beta.$$

Discussion Questions:

1. Converting the matrix $[T]_\alpha^\alpha$ to the matrix $[T]_{\alpha'}^{\alpha'}$ will require using change of basis matrices. Based on Proposition 3, determine which change of basis matrices to apply on which sides of $[T]_\alpha^\alpha$, to obtain $[T]_{\alpha'}^{\alpha'}$. Call your resulting equation Proposition 4.

2. The matrix $[I]_\alpha^{\alpha'}$ is explicitly related to the matrix $[I]_{\alpha'}^{\alpha}$. Use Proposition 3 to prove that these two matrices are in fact inverses of each other.

3. Suppose now that $V = \mathbb{R}^3$, and $\alpha = \{(1,0,1), (0,1,1), (1,1,0)\}$, and $\alpha' = \varepsilon$. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined by

$$[T]_\varepsilon^\varepsilon = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}.$$

   Compute the matrix $[T]_\alpha^\alpha$, by

   (a) determining the change of basis matrices explicitly, using Proposition 2 and your answer from question 2. You will need to apply the *Gauss-Jordan method* for finding the inverse of a matrix.

   (b) computing the matrix product given in your equation labelled Proposition 4.