Linear Algebra, Spring 2004 **Prof. Frechette**

Group Discussion # 2: Mappings on Vector Spaces

Solutions due: Friday, February 27th, at the *beginning* of class
One write-up per group please, **following the same guidelines as the homework.**

Background:

Now that we've developed the basics about vector spaces, it's time to start operating on them (minus the sharp metal implements). We'll spend the next few weeks investigating mappings $T : V \to W$, where V and W are vector spaces. Recall that with groups, we found that the "nice" mappings were the *homomorphisms*, because they had some desirable properties in terms of the group operation. What does "nice" mean for vector space mappings? We're after properties in terms of the vector sum and scalar multiplication operations this time. In this discussion, you'll figure out how we should define "nice" in this setting!

Discussion Questions:

1. Let $V = W = \mathbb{R}^2$ and consider the *rotation mapping* $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined such that for each $\vec{v} \in \mathbb{R}^2$, $R_\theta(\vec{v})$ is obtained by rotating $\vec{v}$ by an angle θ counterclockwise about the origin.
   (a) When $\theta = \pi/2$, explain why $R_{\pi/2}((1,0)) = (0,1)$ and $R_{\pi/2}((0,1)) = (-1,0)$. What is $R_{\pi/2}((-1/2, \sqrt{3}/2))$?
   (b) Show that $R_{\pi/2}((x,y)) = (-y,x)$ for every $(x,y) \in \mathbb{R}^2$.
   (c) Now look at linear combinations instead of single vectors. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ and let $c \in \mathbb{R}$. Compute $R_{\pi/2}(c\vec{u} + \vec{v})$. How does this relate to $R_{\pi/2}(\vec{u})$ and $R_{\pi/2}(\vec{v})$?
   (d) Now for the general case: the rotation mapping R_θ is defined by
   $$R_\theta((x,y)) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$
   As above, evaluate $R_\theta(c\vec{u} + \vec{v})$, and determine how this relates to $R_\theta(\vec{u})$ and $R_\theta(\vec{v})$.

2. Next, let $V = \mathbb{R}^3$, and $W = \mathbb{R}^2$, and consider the mapping $L : V \to W$ defined by

$$L((x, y, z)) = (x + 2y, 3x + 4z).$$

   (a) What are $L(\vec{e_1})$, $L(\vec{e_2})$, and $L(\vec{e_3})$, where $\vec{e_i}$ are the standard basis vectors in $\mathbb{R}^3$?

   (b) Let $\vec{u}, \vec{v} \in \mathbb{R}^3$, and let $c \in \mathbb{R}$. Compute $L(c\vec{u} + \vec{v})$, and determine how the result relates to $L(\vec{u})$ and $L(\vec{v})$.

3. Next, let $V = C^1(\mathbb{R})$, the space of all continuously differentiable functions. (Thus $f \in V$ means f' exists and is continuous on $\mathbb{R}$.) Let $W = C(\mathbb{R})$. Consider the mapping $D : V \to W$ defined by $D(f) = f'$. (Thus D maps a function f to its deriative function f'.)

   (a) What is $D(5e^{7x}+\cos(3x))$? How does this result relate to $D(e^{7x})$ and $D(\cos(3x))$?

   (b) In general, for any $f, g \in V$ and any $c \in \mathbb{R}$, how does $D(cf + g)$ relate to $D(f)$ and $D(g)$?

4. As our last example, let $V = C(\mathbb{R})$ and $W = \mathbb{R}$. Define $I : V \to W$ by

$$I(f) = \int_0^1 f(x)\, dx.$$

   (Yes folks, there is an integral in your algebra class.) Answer the same questions that you answered in question 3, with the mapping D replaced by the mapping I.

5. Now the moment of truth! What patterns did you observe in the last parts of each of the questions above? You should now have a conjecture for the "nice" property for mappings on vector spaces that we wish to study. State the property mathamtically, and then explain it in words. Explain one property about this kind of mapping that makes it desirable. (Use the idea of *group homomorphism* for comparison.)