A Crystal Definition for Symplectic Multiple Dirichlet Series

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April 30, 2013

1 Definition of the Multiple Dirichlet Series

In this section, we present general notation for root systems and the corresponding Weyl group multiple Dirichlet series.

1.1 Root Systems

Let Φ be a reduced root system contained in V, a real vector space of dimension r. The dual vector space $V^\vee$ contains a root system $\Phi^\vee$ in bijection with Φ, where the bijection switches long and short roots. If we write the dual pairing

$$V \times V^\vee \longrightarrow \mathbb{R} : \quad (x, y) \mapsto B(x, y), \tag{1}$$

then $B(\alpha, \alpha^\vee) = 2$. Moreover, the simple reflection $\sigma_\alpha : V \to V$ corresponding to α is given by

$$\sigma_\alpha(x) = x - B(x, \alpha^\vee)\alpha.$$

Note that σ_α preserves Φ. Similarly, we define $\sigma_{\alpha^\vee} : V^\vee \to V^\vee$ by $\sigma_{\alpha^\vee}(x) = x - B(\alpha, x)\alpha^\vee$ with $\sigma_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

For our purposes, without loss of generality, we may take Φ to be irreducible (i.e., there do not exist orthogonal subspaces Φ_1, Φ_2 with $\Phi_1 \cup \Phi_2 = \Phi$). Then set $\langle \cdot, \cdot \rangle$ to be the Euclidean inner product on V and $||\alpha|| = \sqrt{\langle \alpha, \alpha \rangle}$ the Euclidean norm, where we normalize so that $2\langle \alpha, \beta \rangle$ and $||\alpha||^2$ are integral for all $\alpha, \beta \in \Phi$. With this notation,

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \quad \text{for any } \alpha, \beta \in \Phi \tag{2}$$

We partition Φ into positive roots Φ^+ and negative roots Φ^- and let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ denote the subset of simple positive roots. Further, we will denote the fundamental dominant weights by ϵ_i for $i = 1, \ldots, r$ satisfying

$$\frac{2\langle \epsilon_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij} \quad \delta_{ij} : \text{Kronecker delta.} \tag{3}$$

Any dominant weight λ is expressible in terms of the ϵ_i, and a distinguished role in the theory is played by the Weyl vector ρ, defined by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{r} \epsilon_i. \tag{4}$$

1.2 Algebraic Preliminaries

In keeping with the foundations used in previous papers (cf. [5] and [6]) on Weyl group multiple Dirichlet series, we choose to define our Dirichlet series as indexed by integers rather than ideals. By using this approach, the coefficients of the Dirichlet series will closely resemble classical exponential sums, but some care needs to be taken to ensure the resulting series remains well-defined up to units.

To this end, we require the following definitions. Given a fixed positive odd integer n, let F be a number field containing the $2n^{\text{th}}$ roots of unity, and let S be a finite set of places containing all ramified places over $\mathbb{Q}$, all archimedean places, and enough additional places so that the ring of S-integers $\mathcal{O}_S$ is a principal ideal domain. Recall that the $\mathcal{O}_S$ integers are defined as

$$\mathcal{O}_S = \{a \in F \ \mid \ a \in \mathcal{O}_v \ \forall v \notin S\},$$

and can be embedded diagonally in

$$F_S = \prod_{v \in S} F_v.$$

There exists a pairing

$$(\cdot, \cdot)_S : F_S^\times \times F_S^\times \longrightarrow \mu_n \text{ defined by } (a, b)_S = \prod_{v \in S} (a, b)_v,$$

where the $(a, b)_v$ are local Hilbert symbols associated to n and v.

Further, to any $a \in \mathcal{O}_S$ and any ideal $\mathfrak{b} \in \mathcal{O}_S$, we may associate the nth power residue symbol $\left(\frac{a}{\mathfrak{b}}\right)_n$ as follows. For prime ideals $\mathfrak{p}$, the expression $\left(\frac{a}{\mathfrak{p}}\right)_n$ is the unique n^{th} root of unity satisfying the congruence

$$\left(\frac{a}{\mathfrak{p}}\right)_n \equiv a^{(N(\mathfrak{p})-1)/n} \pmod{\mathfrak{p}}.$$

We then extend the symbol to arbitrary ideals $\mathfrak{b}$ by multiplicativity, with the convention that the symbol is 0 whenever a and $\mathfrak{b}$ are not relatively prime. Since $\mathcal{O}_S$ is a principal ideal domain by assumption, we will write

$$\left(\frac{a}{b}\right)_n = \left(\frac{a}{\mathfrak{b}}\right)_n \quad \text{for } \mathfrak{b} = b\mathcal{O}_S$$

and often drop the subscript n on the symbol when the power is understood from context.

Then if a, b are coprime integers in $\mathcal{O}_S$, we have the nth power reciprocity law (cf. [22], Thm. 6.8.3)

$$\left(\frac{a}{b}\right) = (b, a)_S \left(\frac{b}{a}\right) \tag{5}$$

which, in particular, implies that if $\epsilon \in \mathcal{O}_S^\times$ and $b \in \mathcal{O}_S$, then

$$\left(\frac{\epsilon}{b}\right) = (b, \epsilon)_S.$$

Finally, for a positive integer t and $a, c \in \mathcal{O}_S$ with $c \neq 0$, we define the Gauss sum $g_t(a, c)$ as follows. First, choose a non-trivial additive character ψ of F_S trivial on the $\mathcal{O}_S$ integers (cf. [2] for details). Then the n^{th}-power Gauss sum is given by

$$g_t(a, c) = \sum_{d \bmod c} \left(\frac{d}{c}\right)_n^t \psi\left(\frac{ad}{c}\right), \tag{6}$$

where we have suppressed the dependence on n in the notation on the left. The Gauss sum g_t is not multiplicative, but rather satisfies

$$g_t(a, cc') = \left(\frac{c}{c'}\right)_n^t \left(\frac{c'}{c}\right)_n^t g_t(a, c) g_t(a, c') \tag{7}$$

for any relatively prime pair $c, c' \in \mathcal{O}_S$.

1.3 Kubota's Rank 1 Dirichlet series

Many of the definitions for Weyl group multiple Dirichlet series are natural extensions of those from the rank 1 case, so we begin with a brief description of these. We will also need the form of these series when demonstrating functional equations for specific examples in Section 3.

A subgroup $\Omega \subset F_S^\times$ is said to be *isotropic* if $(a,b)_S = 1$ for all $a, b \in \Omega$. In particular, $\Omega = \mathcal{O}_S(F_S^\times)^n$ is isotropic (where $(F_S^\times)^n$ denotes the n^{th} powers in $F_S^\times$). Let $\mathcal{M}_t(\Omega)$ be the space of functions $\Psi : F_S^\times \longrightarrow \mathbb{C}$ that satisfy the transformation property

$$\Psi(\epsilon c) = (c, \epsilon)_S^{-t}\Psi(c) \quad \text{for any } \epsilon \in \Omega, c \in F_S^\times. \tag{8}$$

For $\Psi \in \mathcal{M}_t(\Omega)$, consider the following generalization of Kubota's Dirichlet series:

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathcal{O}_s/\mathcal{O}_s^\times} \frac{g_t(a,c)\Psi(c)}{|c|^{2s}}. \tag{9}$$

Here $|c|$ is the order of $\mathcal{O}_S/c\mathcal{O}_S$, $g_t(a,c)$ is as in (6) and the term $g_t(a,c)\Psi(c)|c|^{-2s}$ is independent of the choice of representative c, modulo S-units. Standard estimates for Gauss sums show that the series is convergent if $\Re(s) > \frac{3}{4}$. Our functional equation computations will hinge on the functional equation for this Kubota Dirichlet series. Before stating this result, we require some additional notation. Let

$$\mathbf{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-2} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{10}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\mathbf{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}_t^*(s, \Psi, a) = \mathbf{G}_m(s)^{[F:\mathbb{Q}]/2} \zeta_F(2ms - m + 1)\mathcal{D}_t(s, \Psi, a), \tag{11}$$

where $m = n/\gcd(n,t)$, $\frac{1}{2}[F : \mathbb{Q}]$ is the number of archimedean places of the totally complex field F, and ζ_F is the Dedekind zeta function of F.

If $v \in S_{fin}$ let q_v denote the cardinality of the residue class field $\mathcal{O}_v/\mathcal{P}_v$, where $\mathcal{O}_v$ is the local ring in F_v and $\mathcal{P}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}. If $\Psi \in \mathcal{M}_t(\Omega)$ and $\eta \in F_S^\times$, denote

$$\widetilde{\Psi}_\eta(c) = (\eta, c)_S\, \Psi(c^{-1}\eta^{-1}). \tag{12}$$

Then we have the following result (Theorem 1 in [6]), which follows from the work of Brubaker and Bump [2].

Theorem 1 *Let $\Psi \in \mathcal{M}_t(\Omega)$ and $a \in \mathcal{O}_S$. Let $m = n/\gcd(n,t)$. Then $\mathcal{D}_t^*(s,\Psi,a)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2m}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta^t(s)$ depending only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}_t^*(s,\Psi,a) = |a|^{1-2s} \sum_{\eta \in F_S^\times/(F_S^\times)^n} P_{a\eta}^t(s) \mathcal{D}_t^*(1-s, \widetilde{\Psi}_\eta, a). \tag{13}$$

This result, based on ideas of Kubota [18], relies on the theory of Eisenstein series. The case $t = 1$ is handled in [2]; the general case follows as discussed in the proof of Proposition 5.2 of [5]. Notably, the factor $|a|^{1-2s}$ is independent of the value of t.

1.4 The form of higher rank multiple Dirichlet series

We now begin explicitly defining the multiple Dirichlet series, retaining our previous notation. By analogy with the rank 1 definition in (8), given an isotropic subgroup Ω, let $\mathcal{M}(\Omega^r)$ be the space of functions $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ that satisfy the transformation property

$$\Psi(\epsilon \mathbf{c}) = \left(\prod_{i=1}^r (\epsilon_i, c_i)_S^{||\alpha_i||^2} \prod_{i<j} (\epsilon_i, c_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \right) \Psi(\mathbf{c}) \tag{14}$$

for all $\epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \Omega^r$ and all $\mathbf{c} = (c_1, \ldots, c_r) \in (F_S^\times)^r$.

Recall from the introduction that, given a reduced root system Φ of fixed rank r, an integer $n \geq 1$, $\mathbf{m} \in \mathcal{O}_S^r$, and $\Psi \in \mathcal{M}(\Omega^r)$, we consider a function of r complex variables $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ of the form

$$Z_\Psi(s_1, \ldots, s_r; m_1, \ldots, m_r) = Z_\Psi(\mathbf{s}; \mathbf{m}) = \sum_{\mathbf{c} = (c_1, \ldots, c_r) \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r} \frac{H^{(n)}(\mathbf{c}; \mathbf{m}) \Psi(\mathbf{c})}{|c_1|^{2s_1} \cdots |c_r|^{2s_r}}.$$

The function $H^{(n)}(\mathbf{c}; \mathbf{m})$ carries the main arithmetic content. It is not defined as a multiplicative function, but rather a "twisted multiplicative" function. For us, this means that for S-integer vectors $\mathbf{c}, \mathbf{c}' \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r$ with $\gcd(c_1 \cdots c_r, c_1' \cdots c_r') = 1$,

$$H^{(n)}(c_1 c_1', \ldots, c_r c_r'; \mathbf{m}) = \mu(\mathbf{c}, \mathbf{c}') H^{(n)}(\mathbf{c}; \mathbf{m}) H^{(n)}(\mathbf{c}'; \mathbf{m}) \tag{15}$$

where $\mu(\mathbf{c}, \mathbf{c}')$ is an n^{th} root of unity depending on $\mathbf{c}, \mathbf{c}'$. It is given precisely by

$$\mu(\mathbf{c}, \mathbf{c}') = \prod_{i=1}^{r} \left(\frac{c_i}{c'_i}\right)_n^{||\alpha_i||^2} \left(\frac{c'_i}{c_i}\right)_n^{||\alpha_i||^2} \prod_{i<j} \left(\frac{c_i}{c'_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{c'_i}{c_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \tag{16}$$

where $\left(\frac{\cdot}{\cdot}\right)_n$ is the n^{th} power residue symbol defined in Section 1.2. Note that in the special case $\Phi = A_1$, the twisted multiplicativity in (15) and (16) agrees with the identity for Gauss sums in (7) in accordance with the numerator for the rank one case given in (9).

Remark 1 We often think of twisted multiplicativity as the appropriate generalization of multiplicativity for the metaplectic group. In particular, for $n = 1$ we reduce to the usual multiplicativity on relatively prime coefficients. Moreover, many of the global properties of the Dirichlet series follow (upon careful analysis of the twisted multiplicativity and associated Hilbert symbols) from local properties, e.g. functional equations as in [5] and [6]. For more on this perspective, see [15].

Note that the transformation property of functions in $\mathcal{M}(\Omega^r)$ in (14) above is motivated by the identity

$$H^{(n)}(\epsilon\mathbf{c}; \mathbf{m})\Psi(\epsilon\mathbf{c}) = H^{(n)}(\mathbf{c}; \mathbf{m})\Psi(\mathbf{c}) \quad \text{for all } \epsilon \in \mathcal{O}_S^r, \mathbf{c}, \mathbf{m} \in (F_S^\times)^r.$$

The proof can be verified using the n^{th} power reciprocity law from Section 1.2.

Now, given any $\mathbf{m}, \mathbf{m}', \mathbf{c} \in \mathcal{O}_S^r$ with $\gcd(m'_1 \cdots m'_r, c_1 \cdots c_r) = 1$, we let

$$H^{(n)}(\mathbf{c}; m_1 m'_1, \ldots, m_r m'_r) = \prod_{i=1}^{r} \left(\frac{m'_i}{c_i}\right)_n^{-||\alpha_i||^2} H^{(n)}(\mathbf{c}; \mathbf{m}). \tag{17}$$

The definitions in (15) and (17) imply that it is enough to specify the coefficients $H^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for any fixed prime p with $l_i = \text{ord}_p(m_i)$ in order to completely determine $H^{(n)}(\mathbf{c}; \mathbf{m})$ for any pair of S-integer vectors $\mathbf{m}$ and $\mathbf{c}$. These prime-power coefficients are described in terms of data from highest-weight representations associated to $(l_1, \cdots, l_r)$ and will be given precisely in Section 2.

1.5 Weyl group actions

In order to precisely state a functional equation for the Weyl group multiple Dirichlet series, we require an action of the Weyl group W of Φ on the complex parameters $(s_1, \ldots, s_r)$. This arises from the linear action of W, realized as the group generated by the simple reflections $\sigma_{\alpha^\vee}$, on $V^\vee$. From the perspective of Dirichlet series, it is more natural to consider this action shifted by $\rho^\vee$, half the sum of the positive co-roots. Then each $w \in W$ induces a transformation $V_{\mathbb{C}}^\vee = V^\vee \otimes \mathbb{C} \to V_{\mathbb{C}}^\vee$ (still denoted by w) if we require that

$$B(w\alpha, w(\mathbf{s}) - \frac{1}{2}\rho^\vee) = B(\alpha, \mathbf{s} - \frac{1}{2}\rho^\vee).$$

We introduce coordinates on $V_{\mathbb{C}}^\vee$ using simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ as follows. Define an isomorphism $V_{\mathbb{C}}^\vee \to \mathbb{C}^r$ by

$$\mathbf{s} \mapsto (s_1, s_2, \ldots, s_r) \quad s_i = B(\alpha_i, \mathbf{s}). \tag{18}$$

This action allows us to identify $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$, and so the complex variables s_i that appear in the definition of the multiple Dirichlet series may be regarded as coordinates in either space. It is convenient to describe this action more explicitly in terms of the s_i and it suffices to consider simple reflections which generate W. Using the action of the simple reflection σ_{α_i} on the root system Φ given in (2) in conjunction with (18) above gives the following:

Proposition 1 *The action of σ_{α_i} on $\mathbf{s} = (s_1, \ldots, s_r)$ defined implicitly in (18) is given by*

$$s_j \mapsto s_j - \frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \left(s_i - \frac{1}{2} \right) \quad j = 1, \ldots, r. \tag{19}$$

In particular, $\sigma_{\alpha_i} : s_i \mapsto 1 - s_i$.

1.6 Normalizing factors and functional equations

The multiple Dirichlet series must also be normalized using Gamma and zeta factors in order to state precise functional equations. Let

$$n(\alpha) = \frac{n}{\gcd(n, ||\alpha||^2)}, \quad \alpha \in \Phi^+.$$

For example, if $\Phi = C_r$ and we normalize short roots to have length 1, this implies that $n(\alpha) = n$ unless α is a long root and n even (in which case $n(\alpha) = n/2$). By analogy with the zeta factor appearing in (11), for any $\alpha \in \Phi^+$, let

$$\zeta_\alpha(\mathbf{s}) = \zeta \left(1 + 2n(\alpha) B(\alpha, \mathbf{s} - \frac{1}{2}\rho^\vee) \right)$$

where ζ is the Dedekind zeta function attached to the number field F. Further, for $\mathbf{G}_n(s)$ as in (10), we may define

$$\mathbf{G}_\alpha(\mathbf{s}) = \mathbf{G}_{n(\alpha)}\left(\frac{1}{2} + B(\alpha, \mathbf{s} - \frac{1}{2}\rho^\vee)\right). \tag{20}$$

Then for any $\mathbf{m} \in \mathcal{O}_S^r$, the normalized multiple Dirichlet series is given by

$$Z_\Psi^*(\mathbf{s};\mathbf{m}) = \left[\prod_{\alpha\in\Phi^+} \mathbf{G}_\alpha(\mathbf{s})\zeta_\alpha(\mathbf{s})\right] Z_\Psi(\mathbf{s},\mathbf{m}). \tag{21}$$

By considering the product over all positive roots, we guarantee that the other zeta and Gamma factors are permuted for each simple reflection $\sigma_i \in W$, and hence for all elements of the Weyl group.

Given any fixed n, $\mathbf{m}$ and root system Φ, we seek to exhibit a definition for $H^{(n)}(\mathbf{c};\mathbf{m})$ (or equivalently, given twisted multiplicativity, a definition of H at prime-power coefficients) such that $Z_\Psi^*(\mathbf{s};\mathbf{m})$ satisfies functional equations of the form:

$$Z_\Psi^*(\mathbf{s};\mathbf{m}) = |m_i|^{1-2s_i} Z_{\sigma_i\Psi}^*(\sigma_i\mathbf{s};\mathbf{m}) \tag{22}$$

for all simple reflections $\sigma_i \in W$. Here, $\sigma_i\mathbf{s}$ is as in (19) and the function $\sigma_i\Psi$, which essentially keeps track of the rather complicated scattering matrix in this functional equation, is defined as in (37) of [6]. As noted in Section 7 of [6], given functional equations of this type, one can obtain analytic continuation to a meromorphic function of $\mathbb{C}^r$ with an explicit description of polar hyperplanes.

2 Definition of the Prime-Power Coefficients

In Section 3 of [1], we gave a precise definition of the p-power coefficients $H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})$ in a multiple Dirichlet series for root systems of type C_r with n odd. The vector $\mathbf{l} = (l_1, l_2, \ldots, l_r)$ appearing in $H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})$ was associated to a dominant integral element for $Sp_{2r}(\mathbb{C})$ of the form

$$\lambda = (l_1 + l_2 + \cdots + l_r, \ldots, l_1 + l_2, l_1) = \sum_{i=1}^r (\ell_i + 1)\epsilon_i, \tag{23}$$

where ϵ_i for $i = 1, \ldots, r$ are the fundamental dominant weights. The contributions to $H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})$ were parametrized by basis vectors of the highest weight representation of highest weight $\lambda + \rho$, where ρ is the Weyl vector for C_r defined in (4), so that

$$\lambda + \rho = (l_1 + l_2 + \cdots + l_r + r, \ldots, l_1 + l_2 + 2, l_1 + 1) =: (L_r, \cdots, L_1). \tag{24}$$

In this section, we give a definition for the p-parts of H in terms of crystal graphs and their associated BZL-patterns, and we demonstrate how this definition matches the one given in terms of GT-patterns. For precise details of the correspondence between BZL-patterns and GT-patterns of type C_r, we refer the reader to [19].

Consider $Sp_{2r}(\mathbb{C})$, with the enumeration of simple roots given by

$$\alpha_1 = 2\epsilon_m, \quad \alpha_2 = \epsilon_1 - \epsilon_2, \quad \ldots, \quad \alpha_r = \epsilon_{r-1} - \epsilon_r, \tag{25}$$

In Section 6 of [19], Littelmann designates this as a “good enumeration,” and explicitly defines C_λ, the convex polytope that arises from using the associated so-called “nice decomposition” of the long element of the associated Weyl group:

$$w_0 = s_1(s_2s_1s_2)(\ldots)(s_{r-1}\ldots s_1\ldots s_{r-1})(s_rs_{r-1}\ldots s_1\ldots s_{r-1}s_r). \tag{26}$$

(**NOTE:** Previously, we chose coordinates so that the simple roots were

$$\alpha_1 = 2\mathbf{e}_1, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_1, \quad \ldots, \quad \alpha_r = \mathbf{e}_r - \mathbf{e}_{r-1}. \tag{27}$$

Do we need to change the assignment of patterns to k-coordinates based on making a different choice, or can we simply claim that there is some suitable change of variables that results in this assignment?)

Following Littelmann [19], we construct a triangle consisting of r centered rows of boxes, with $2(r+1-i)-1$ entries in the row i, starting from the top. To each vector $\mathbf{c} \in \mathbb{R}^{r^2}$, let $\Delta(\mathbf{c})$ denote the triangle whose entries are the coordinates of $\mathbf{c}$, with the boxes filled from bottom row to top row, and from left to right. We then identify $\mathbf{c}$ with its triangle, written as $\Delta(\mathbf{c}) = (c_{i,j})$, where as Littelmann's notation, $c_{i,j}$ is the j-th entry in the i-th row, but with $i \leq j \leq 2r - i$. Also, for convenience in the discussion below, we will write $\overline{c}_{i,j} := c_{2r-j}$ for $i \leq j \leq r$. Thus when $r = 3$, we are considering triangles of the form

$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$\overline{c}_{1,2}$	$\overline{c}_{1,1}$
	$c_{2,2}$	$c_{2,3}$	$\overline{c}_{2,2}$	
		$c_{3,3}$		

Need to describe how the desired BZL patterns arise from the crystal graph, using the operators f_i (or e_i?).

The following summarizes the construction of $C_{\lambda+\rho}$, as given in Theorem 6.1 and Corollary 1 in Section 6 of [19], with the specialization $\lambda_i = \ell_i + 1$, for $1 \leq i \leq r$.

Proposition 2 *Given the good enumeration w_0 as in (26) for the Weyl group of $Sp_{2r}(\mathbb{C})$, and λ as in (23), then $C_{\lambda+\rho}$ is the convex polytope of all triangles $\Delta(c_{i,j})$ such that the entries in the rows are non-negative and weakly increasing, and satisfy the following upper-bound inequalities for all $1 \leq i \leq r$ and $1 \leq j \leq r-1$:*

$$\begin{aligned} \overline{c}_{i,j} &\leq \ell_{r-j+1} + 1 + s(\overline{c}_{i,j-1}) - 2s(c_{i-1,j}) + s(c_{i-1,j+1}), && (28)\\ c_{i,j} &\leq \ell_{r-j+1} + 1 + s(\overline{c}_{i,j-1}) - 2s(\overline{c}_{i,j}) + s(c_{i,j+1}), && (29)\\ \text{and} \quad c_{i,r} &\leq \ell_1 + 1 + s(\overline{c}_{i,r-1}) - s(c_{i-1,r}). && (30) \end{aligned}$$

Recall from [1], a GT-pattern P has the form

$$P = \begin{array}{ccccccccc} a_{0,1} & & a_{0,2} & & \cdots & & a_{0,r} & & \\ & b_{1,1} & & b_{1,2} & \cdots & b_{1,r-1} & & b_{1,r} & \\ & & a_{1,2} & & \cdots & & a_{1,r} & & \\ & & & \ddots & & \ddots & & \vdots & \\ & & & & & & a_{r-1,r} & & \\ & & & & & & & b_{r,r} & \end{array} \qquad (31)$$

where the $a_{i,j}, b_{i,j}$ are non-negative integers and the rows of the pattern interleave. That is, for all $a_{i,j}, b_{i,j}$ in the pattern P above,

$$\min(a_{i-1,j}, a_{i,j}) \geq b_{i,j} \geq \max(a_{i-1,j+1}, a_{i,j+1})$$

and

$$\min(b_{i+1,j-1}, b_{i,j-1}) \geq a_{i,j} \geq \max(b_{i+1,j}, b_{i,j}).$$

We considered the set of all GT-patterns with top row $(a_{0,1}, \ldots, a_{0,r}) = (L_r, \ldots, L_1)$, which form a basis for the highest weight representation with highest weight $\lambda + \rho$, and referred to this set of patterns as $GT(\lambda + \rho)$.

Proposition 3 (DEFINE) *Map between GT-patterns and BZL-patterns...*

Remark 2 In Section 6 of [19], Littelmann gives an example illustrating this correspondence, in the case of rank 3. This example is given below, with the corrected first entry in the second row.

$$\begin{array}{cccccc} 9 & & 5 & & 1 & \\ & 6 & & 5 & & 0 \\ & & 5 & & 3 & \\ & & & 5 & & 2 \\ & & & & 3 & \\ & & & & & 1 \end{array} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|c|} \hline 7 & 7 & 4 & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 1 & 0 & \multicolumn{1}{|c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

We will subject our BZL-patterns to certain *decoration rules* that will be used to explicitly determine each pattern's contribution to the sum defining the p-power coefficients of our multiple Dirichlet series. These decorations will consist of *boxes* and *circles*, applied according to the following rules:

1. The entry $c_{i,j}$ is circled if $c_{i,j} = c_{i,j+1}$. We understand the entries outside the triangular array to be zeroes, so the right-most entry in a row will be circled if it equals 0.

2. The entry $c_{i,j}$ is boxed if equality holds in the upper-bound inequality for $c_{i,j}$ given above in Proposition 2.

The above map between GT-patterns and BZL-patterns, together with the decoration rules, is illustrated in the following example.

$$\begin{array}{cccccc} 5 & & 3 & & 1 & \\ & 4 & & 2 & & 1 \\ & & 4 & & 2 & \\ & & & 3 & & 2 \\ & & & & 3 & \\ & & & & & 1 \end{array} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|c|} \hline \boxed{5} & \boxed{3} & 2 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 1 & 1 & \multicolumn{1}{|c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

The contributions to each $H^{(n)}(p^{\mathbf{k}}; p^{\mathbf{l}})$ with both $\mathbf{k}$ and $\mathbf{l}$ fixed come from a single weight space corresponding to $\mathbf{k} = (k_1, \ldots, k_r)$ in the highest weight representation $\lambda + \rho$ corresponding to $\mathbf{l}$. Given a BZL-pattern $\Delta(c_{i,j})$, define the vector $k(\Delta) = (k_1(\Delta), k_2(\Delta), \ldots, k_r(\Delta))$ with

$$\begin{aligned} k_1(\Delta) &= \sum_{j=1}^{r} c_{i,r}, \\ \text{and} \quad k_i(\Delta) &= \sum_{j=1}^{r+1-i} (c_{j,r+1-i} + \overline{c}_{j,r+1-i}), \quad \text{for } 1 < i \le r. \end{aligned} \tag{32}$$

We define

$$H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}}) = H^{(n)}(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r}) = \sum_{\substack{\Delta\in\mathcal{C}_{\lambda+\rho}\\ k(\Delta)=(k_1,\ldots,k_r)}} G(\Delta) \tag{33}$$

where the sum is over all BZL-patterns Δ with top row $(L_r,\ldots,L_1)$ as in (24) satisfying the condition $\mathbf{k}(\Delta)=(k_1,\ldots,k_r)$ and $G(\Delta)$ is a weighting function whose definition which is defined as follows: To each entry $c_{i,j}$ in Δ, we associate

$$\gamma(c_{i,j}) = \begin{cases} q^{c_{i,j}} & \text{if } c_{i,j} \text{ is circled (but not boxed),} \\ g_{\delta_{jr}+1}(p^{c_{i,j}-1},p^{c_{i,j}}) & \text{if } c_{i,j} \text{ is boxed (but not circled),} \\ \phi(p^{c_{i,j}}) & \text{if neither,} \\ 0 & \text{if both.} \end{cases} \tag{34}$$

where $g_t(p^\alpha,p^\beta)$ is an n^{th}-power Gauss sum as in (6), $\phi(p^a)$ denotes Euler's totient function for $\mathcal{O}_S/p^a\mathcal{O}_S$, $q=|\mathcal{O}_S/p\mathcal{O}_S|$ and δ_{jr} is the Kronecker delta function. We then define

$$G(\Delta) = \prod_{\substack{1\le i\le r,\\ i\le j\le 2r-1}} \gamma(c_{i,j}). \tag{35}$$

3 Functional equations by reduction to rank 1

In this section, we provide evidence toward global functional equations for the multiple Dirichlet series $Z_\Psi(\mathbf{s};\mathbf{m})$ through a series of computations in a particular rank 2 example. We will demonstrate that these multiple Dirichlet series are, in some sense, built from combinations of rank 1 Kubota Dirichlet series and thus inherit their functional equations. Similar techniques to those presented here would apply for arbitrary rank.

Recall from (19) that, in rank 2, we expect functional equations corresponding to the simple reflections

$$\sigma_1 : (s_1,s_2) \mapsto (1-s_1, s_1+s_2-1/2) \text{ and } \sigma_2 : (s_1,s_2)\mapsto (s_1+2s_2-1, 1-s_2), \tag{36}$$

which generate a group acting on $(s_1,s_2)\in\mathbb{C}^2$ isomorphic to the Weyl group of C_2, the dihedral group of order 8.

With notations as before, let $n=3$, and $\mathbf{m}=(p^2,p^1)$ for some fixed $\mathcal{O}_S$ prime p. Then we will illustrate how our definition of the coefficients $H^{(3)}(\mathbf{c};p^2,p)$ leads to a multiple Dirichlet series $Z_\Psi(\mathbf{s};p^2,p)$ satisfying the functional equations

$$Z_\Psi(s_1,s_2;p^2,p) \to |p^2|^{1-2s_1} Z_{\sigma_1\Psi}(1-s_1,s_1+s_2-1/2;p^2,p) \tag{37}$$

and

$$Z_\Psi(s_1,s_2;p^2,p) \to |p|^{1-2s_2} Z_{\sigma_2\Psi}(s_1+2s_2-1,1-s_2;p^2,p) \tag{38}$$

corresponding to the above simple reflections according to (22).

Our strategy is quite simple. To demonstrate the functional equation corresponding to σ_1, write

$$Z_\Psi(s_1,s_2;p^2,p) = \sum_{c_2\in\mathcal{O}_S/\mathcal{O}_S^\times} |c_2|^{-2s_2} \sum_{c_1\in\mathcal{O}_S/\mathcal{O}_S^\times} \frac{H^{(3)}(c_1,c_2;p^2,p)\Psi(\mathbf{c})}{|c_1|^{2s_1}} \tag{39}$$

and attempt to realize the inner sum, for any fixed c_2, in terms of rank 1 Kubota Dirichlet series whose one-variable functional equations are all compatible with the global functional equation in (37). Similar methods apply for the other simple reflection. One difficulty with this approach is that our definitions for $H^{(n)}(\mathbf{c};\mathbf{m})$ up to this point have been "local" – that is, we have only provided explicit definitions for the prime power supported coefficients. Of course, our requirement that the $H^{(n)}(\mathbf{c};\mathbf{m})$ satisfy twisted multiplicativity then uniquely defines the coefficients for any r-tuple of integers $\mathbf{c}$, but there are many complications in attempting to patch together the prime-power supported pieces to reconstruct a global series.

This strategy was precisely carried out in [5] and [6] for any root system Φ provided n satisfies the Stability Assumption stated in (**??**). Indeed, global objects were reconstructed from the prime-power supported contributions by meticulously checking that all Hilbert symbols and n^{th} power residue symbols combine neatly into Kubota Dirichlet series with the required twisted multiplicativity. Our purpose here is not to get bogged down in these complications, but rather to show how global functional equations can be anticipated simply by considering the prime-power supported coefficients. Note that in the example at hand, the stable cases for $\mathbf{m} = (p^2,p)$ require $n \geq 7$, so $n = 3$ is not stable and the results of [6] do not apply. Nevertheless, as we will explain, our method of reduction to the rank 1 case is still viable.

3.1 Analysis of $H^{(3)}(c_1,c_2;p^2,p)$ with prime-power support

The nature of $H^{(3)}(c_1,c_2;p^2,p)$ with c_1,c_2 powers of a fixed prime depends critically on whether that prime is p, the fixed prime occurring in $\mathbf{m} = (p^2,p)$, or a distinct prime $\ell \neq p$. The prime-power supported coefficients $H^{(3)}(\ell^{k_1},\ell^{k_2};p^2,p)$ at primes $\ell \neq p$ have identical support (k_1,k_2) for any such prime ℓ (as the support depends only on $\mathrm{ord}_\ell(m_1)$ and $\mathrm{ord}_\ell(m_2)$) and a uniform description as products of Gauss sums in terms of ℓ. The (k_1,k_2) coordinates of this support are depicted in Figure 1

– the result of the affine linear transformation of the weights in the corresponding highest weight representation ρ. The vertex in the bottom left corner is placed at $(k_1, k_2) = (0, 0)$. At each of the vertices in the interior, the number shown indicates the number of BZL-patterns associated with that vertex, that is, the multiplicity in the associated weight space. These counts include both strict and non-strict patterns, though non-strict patterns give no contribution to the multiple Dirichlet series for any n. Support on the boundary is indicated by black dots, each with a unique corresponding BZL-pattern.

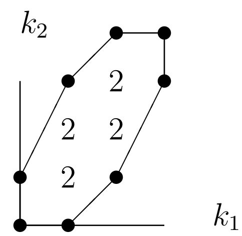

Figure 1: Support (k_1, k_2) for $H^{(3)}(\ell^{k_1}, \ell^{k_2}; p^2, p)$ (with indicated multiplicities of contributing BZL patterns P having $k(P) = (k_1, k_2)$).

For $n = 3$ each of the 8 patterns P (4 strict, 4 non-strict) in the interior of the polygon of support have $G(P) = 0$, so the only non-zero contributions come from the 8 boundary vertices. Note that these are just the "stable" vertices, which have $G(P)$ non-zero for all n.

The coefficients $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p^1)$ are much more interesting. Recall these coefficients are parametrized by BZL-patterns with top row $(L_2, L_1) = (5, 3)$ according to (24). **NEEDS TO BE CHANGED.** The supporting vertices (k_1, k_2) for the p-part are shown below in Figure 2. On the support's boundary, stable vertices are indicated by filled circles and unstable vertices are indicated by open circles, all with multiplicity one.

Again, the choice of $n = 3$ will make $G(\Delta) = 0$ for many of the patterns Δ occurring at these support vertices. Roughly speaking, the non-zero support for any fixed n forms an $n \times n$ regular lattice beginning at the origin. However, this lattice becomes somewhat distorted by the boundary of the polygon, particularly the location of the stable vertices. In fact, our choice of $(l_1, l_2) = (2, 1)$ in this example is so small that this phenomenon is essentially obscured.

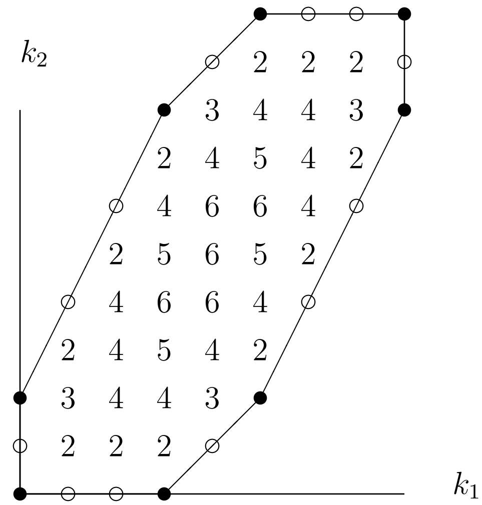

Figure 2: Support (k_1, k_2) for $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ (with indicated multiplicities of contributing BZL patterns Δ).

3.2 Three Specific Examples

Returning to the discussion of functional equations, we will first demonstrate a functional equation corresponding to the simple reflection σ_1 taking $s_1 \mapsto 1 - s_1$. Recall our strategy is to show that for any choice of c_2, we may write the inner sum in (39) in terms of Kubota Dirichlet series. For example, let $c_2 = p^8$. By twisted multiplicativity, we see that $H^{(3)}(c_1, p^8; p^2, p)$ will be 0 unless $\mathrm{ord}_\ell(c_1) \leq 1$ for all primes $\ell \neq p$ (as evident from Figure 1, since we seek ℓ-power terms with support $k_2 = 0$). More interestingly, using Figure 2, we see that p-power terms with $k_2 = 8$ must have $3 \leq \mathrm{ord}_p(c_1) \leq 8$. Let's examine the p-power coefficients more closely.

3.2.1 The functional equation σ_1 with $k_2 = 8$.

As seen in Figure 2, $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ with $k_2 = 8$ has support at 6 lattice points $(k_1, 8)$ with a total of 16 BZL-patterns. Having chosen $n = 3$ (so that all Gauss sums appearing are formed with a cubic residue symbol), one checks that only five of these 16 BZL-patterns have non-zero Gauss sum products associated to them. These are listed in the table below.

Δ	$k(\Delta)$	$G(\Delta)$	$G(\Delta)$ for $n=3$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & 0 \\ & 0 & \end{array}$	$(3,8)$	$g_2(p^2,p^3)\, g_1(p^7,p^8)$	$-\lvert p\rvert^2 g_1(p^7,p^8)$
$\begin{array}{ccc} 6 & \boxed{5} & \boxed{2} \\ & 0 & \end{array}$	$(5,8)$	$g_1(p^1,p^2)\, g_2(p^4,p^5)\, g_1(p^7,p^6)$	$\lvert p\rvert^6 \phi(p^6)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & 0 \\ & 3 & \end{array}$	$(6,8)$	$g_2(p^2,p^3)\, g_1(p^7,p^8)\, g_2(p^4,p^3)$	$-\lvert p\rvert^2 g_1(p^7,p^8)\phi(p^3)$
$\begin{array}{ccc} 6 & \boxed{5} & \boxed{2} \\ & \boxed{1} & \end{array}$	$(6,8)$	$g_1(p^1,p^2)\, g_2(p^4,p^5)\, g_1(p^7,p^6)\, g_2(1,p)$	$\lvert p\rvert^6 \phi(p^6) g_2(1,p)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & 0 \\ & \boxed{5} & \end{array}$	$(8,8)$	$g_2(p^2,p^3)\, g_1(p^7,p^8)\, g_2(p^4,p^5)$	$-\lvert p\rvert^2 g_1(p^7,p^8) g_2(p^4,p^5)$

We have computed the final column in the table from the third column, using the following three elementary properties of n^{th}-order Gauss sums at prime powers, which can be proved easily from the definition in (6):

1. If $a \geq b$, then $g_t(p^a,p^b) = \begin{cases} \phi(p^b) & n \mid tb, \\ 0 & n \nmid tb. \end{cases}$

2. For any integers a and t, $g_t(p^{a-1},p^a) = |p|^{a-1} g_{at}(1,p)$.

3. For any integer t, $g_t(1,p)\, g_{n-t}(1,p) = |p|$.

For notational convenience, let the inner sum in (39) be denoted

$$F(s_1;c_2) = \sum_{c_1 \in \mathcal{O}_S/\mathcal{O}_S^\times} \frac{H^{(3)}(c_1,c_2;p^2,p)\Psi(\mathbf{c})}{|c_1|^{2s_1}}. \tag{40}$$

Fix $c_2 = p^8$ and let

$$F^{(p)}(s_1; p^8) = \sum_{k_1} \frac{H^{(3)}(p^{k_1}, p^8; p^2, p)\Psi(p^{k_1}, p^8)}{|p|^{2k_1 s_1}}. \tag{41}$$

From the table above, this sum is supported at $k_1 = 3, 5, 6$ and 8, so that $F^{(p)}(s_1; p^8)$ equals

$$\begin{aligned} &\frac{-|p|^2\, g_1(p^7, p^8)\Psi(p^3, p^8)}{|p|^{6s_1}} \left[1 + \frac{g_2(p^4, p^3)}{|p|^{6s_1}} \frac{\Psi(p^6, p^8)}{\Psi(p^3, p^8)} + \frac{g_2(p^4, p^5)}{|p|^{10s_1}} \frac{\Psi(p^8, p^8)}{\Psi(p^3, p^8)}\right] \\ &\qquad + \quad \frac{|p|^6 \phi(p^6)\Psi(p^5, p^8)}{p^{10s_1}} \left[1 + \frac{g_2(1, p)}{|p|^{2s_1}} \cdot \frac{\Psi(p^6, p^8)}{\Psi(p^5, p^8)}\right] \end{aligned} \tag{42}$$

Ignoring complications from the Ψ function, both bracketed sums may be expressed as the p-part of a Kubota Dirichlet series in s_1. Indeed, letting $\mathcal{D}_2^{(p)}$ denote the prime-power supported coefficients of the Kubota Dirichlet series $\mathcal{D}_2$ in (9), then

$$\mathcal{D}_2^{(p)}(s_1, \Psi', p^4) = \left[1 + \frac{g_2(p^4, p^3)}{|p|^{6s_1}} \frac{\Psi(p^6, p^8)}{\Psi(p^3, p^8)} + \frac{g_2(p^4, p^5)}{|p|^{10s_1}} \frac{\Psi(p^8, p^8)}{\Psi(p^3, p^8)}\right]$$

for some appropriately defined $\Psi' \in \mathcal{M}_2(\Omega)$, as $\mathcal{D}_2^{(p)}(s_1, \Psi', p^4)$ contains $g_2(p^4, p^{k_1})$ in the numerator, which is non-zero only if $k_1 = 0, 3$ or 5 when $n = 3$. Similarly,

$$\mathcal{D}_2^{(p)}(s_1, \Psi'', 1) = \left[1 + \frac{g_2(1, p)}{|p|^{2s_1}} \cdot \frac{\Psi(p^6, p^8)}{\Psi(p^5, p^8)}\right]$$

for an appropriately defined $\Psi'' \in \mathcal{M}_2(\Omega)$. Thus, according to (42), we may express $F^{(p)}(s_1)$ as the sum of p-parts of Kubota Dirichlet series multiplied by Dirichlet monomials. The reader interested in checking all details regarding the Ψ function should refer to Section 5 of [5]; our notation for the one-variable Ψ' or Ψ'' in $M_2(\Omega)$ derived from $\Psi(c_1, c_2)$ is called Ψ^{c_1, c_2} in Lemma 5.3 of [5].

In order to reconstruct the global object $F(s_1; c_2)$ with $c_2 = p^8$, we now turn to the analysis at primes $\ell \neq p$. Since $\mathrm{ord}_\ell(c_2) = 0$, then we can reconstruct $F(s_1; p^8)$ from the twisted multiplicativity in (15) and (17) together with knowledge of terms of the form $H^{(3)}(\ell^{k_1}, 1; p^2, p)$. Then define

$$F^{(\ell)}(s_1; 1) = \sum_{k_1} \frac{H^{(3)}(\ell^{k_1}, 1; p^2, p)\Psi(\ell^{k_1}, p^8)}{|\ell|^{2k_1 s_1}}$$

for all primes $\ell \neq p$. Using twisted multiplicativity in (17),

$$\begin{aligned} F^{(\ell)}(s_1; 1) &= \sum_{k_1} \left(\frac{p^2}{\ell^{k_1}}\right)_3^{-2} \left(\frac{p}{1}\right)_3^{-1} \frac{H^{(3)}(\ell^{k_1}, 1; 1, 1)\Psi(\ell^{k_1}, p^8)}{|\ell|^{2k_1 s_1}} \\ &= \Psi(1, p^8) + \left(\frac{p^2}{\ell}\right)_3^{-2} H^{(3)}(\ell^1, 1; 1, 1)\Psi(\ell^1, p^8)|\ell|^{-2s_1} \\ &= \Psi(1, p^8)\left[1 + \left(\frac{p^2}{\ell}\right)_3^{-2} \frac{g_2(1, \ell)}{|\ell|^{2s_1}} \frac{\Psi(\ell^1, p^8)}{\Psi(1, p^8)}\right]. \end{aligned}$$

To summarize, we have found that

$$F^{(p)}(s_1; p^8) = \frac{-|p|^2\, g_1(p^7, p^8)\Psi(p^3, p^8)}{|p|^{6s_1}} \mathcal{D}_2^{(p)}(s_1, \Psi', p^4) + \\ \frac{|p|^6 \phi(p^6)\Psi(p^5, p^8)}{|p|^{10s_1}} \mathcal{D}_2^{(p)}(s_1, \Psi'', 1)$$

and

$$F^{(\ell)}(s_1; 1) = \Psi(1, p^8)\left[1 + \left(\frac{p^2}{\ell}\right)_3^{-2} \frac{g_2(1, \ell)}{|\ell|^{2s_1}} \frac{\Psi(\ell^1, p^8)}{\Psi(1, p^8)}\right], \text{ for all primes } \ell \neq p.$$

Now using twisted multiplicativity, we can reconstruct $F(s_1; p^8)$. We claim that

$$F(s_1; p^8) = \frac{-|p|^2\, g_1(p^7, p^8)\Psi(p^3, p^8)}{|p|^{6s_1}} \mathcal{D}_2(s_1, \Psi', p^4) + \frac{|p|^6 \phi(p^6)\Psi(p^5, p^8)}{|p|^{10s_1}} \mathcal{D}_2(s_1, \Psi'', 1).$$

This may be directly verified up to Hilbert symbols (i.e. ignoring Hilbert symbols in the power reciprocity law in (5)) by using twisted multiplicativity to reconstruct $H(c_1, p^8; p^2, p)$ from $F^{(p)}(s_1; p^8)$ and $F^{(\ell)}(s_1; 1)$. But to give a full accounting with Hilbert symbols one needs to verify that the "left-over" Hilbert symbols from repeated applications of reciprocity are precisely those required for the definitions of Ψ' and Ψ'' (again referring to Lemma 5.3 of [5]).

We now return to our general strategy of demonstrating the functional equation σ_1 as in (36). The function $Z_\Psi(s_1, s_2; p^2, p)$ as in (39) with fixed $c_2 = p^8$ yields $F(s_1; p^8)$ as above. We must verify that this portion of $Z_\Psi(s_1, s_2; p^2, p)$ is consistent with the desired global functional equation

$$Z_\Psi(s_1, s_2; p^2, p) \to |p^2|^{1-2s_1} Z_{\sigma_1\Psi}(1 - s_1, s_1 + s_2 - 1/2; p^2, p)$$

presented at the outset of this section. By Theorem 1,

$$\mathcal{D}_2(s_1, \Psi', p^4) \to |p^4|^{1-2s_1} \mathcal{D}_2(1-s_1, \Psi', p^2)$$

and $|p|^{-6s_1-16s_2} \to |p|^{2-10s_1-16s_2}$ under σ_1. Similarly, $\mathcal{D}_2(s_1, \Psi'', 1) \to \mathcal{D}_2(1-s_1, \Psi'', p^2)$ and $|p|^{-10s_1-16s_2} \to |p|^{-2-6s_1-16s_2}$ under σ_1. Taken together, these calculations imply that

$$\frac{F(s_1; p^8)}{|p^8|^{2s_2}} \to |p^2|^{1-2s_1} \frac{F(1-s_1; p^8)}{|p^8|^{2(s_1+s_2-1/2)}},$$

which is consistent with the global functional equation for Z_Ψ above.

Throughout the above analysis, we chose to restrict to the case where $c_2 = p^8$ to limit the complexity of the calculation. However, identical methods could be used to determine the global object for arbitrary choice of c_2 depending on the order of p dividing c_2, and hence verify the global functional equation for σ_1 in full generality.

Remark 3 With respect to the s_1 functional equation, it turns out to be quite simple to figure out which BZL-patterns contribute to a particular Kubota Dirichlet series appearing in $F(s_1; p^{k_2})$. All such BZL-patterns agree at every entry except the bottom one (which decrements as we increase k_1) as can be verified in our earlier table with $k_2 = 8$. However, as we will see in the next section, functional equations in s_2 and the respective Kubota Dirichlet series used in asserting them obey no such simple pattern.

3.2.2 The functional equation σ_2 with $k_1 = 3$.

We now repeat the methods of the previous section to demonstrate a functional equation under σ_2. As we will show, it is significantly more difficult to organize the local contributions into linear combinations of Kubota Dirichlet series in terms of s_2. Once this is accomplished, the analysis proceeds along the lines of the previous section, so we omit further details.

Let $c_1 = p^3$ be fixed. Mimicking our notation from the previous section, we now set

$$F(s_2; p^3) = \sum_{k_2} \frac{H^{(3)}(p^3, c_2; p^2, p)\Psi(p^3, c_2)}{|c_2|^{2s_2}}. \tag{43}$$

As in the previous section, the bulk of the difficulty lies in analyzing

$$F^{(p)}(s_2; p^3) = \sum_{k_2} \frac{H^{(3)}(p^3, p^{k_2}; p^2, p)\Psi(p^3, p^{k_2})}{|p|^{2k_2 s_2}}.$$

Again referring to Figure 2, coefficients $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ with $k_1 = 3$ involve 9 different vertices and a total of 30 BZL-patterns, only six of which give nonzero contributions in the case when $n = 3$. In the table below, we list only those BZL-patterns yielding nonzero Gauss sums. The final column has again been computed from the third column, using the elementary properties of n^{th}-order Gauss sums mentioned in the previous subsection.

Δ	$(k_1, k_2) = k(\Delta)$	$G(\Delta)$	$G(\Delta)$ for $n = 3$
$\begin{array}{ccc} 0 & 0 & 0 \\ & \boxed{3} & \end{array}$	$(3, 0)$	$g_2(p^2, p^3)$	$-\lvert p\rvert^2$
$\begin{array}{ccc} \boxed{2} & 0 & 0 \\ & 3 & \end{array}$	$(3, 2)$	$g_1(p, p^2)\, g_2(p^4, p^3)$	$\lvert p\rvert g_2(1, p)\phi(p^3)$
$\begin{array}{ccc} 3 & 3 & \boxed{2} \\ & 0 & \end{array}$	$(3, 5)$	$p^3\, g_1(p, p^2)\, g_1(p^4, p^3)$	$\lvert p\rvert^4 g_2(1, p)\phi(p^3)$
$\begin{array}{ccc} \boxed{4} & 3 & \boxed{2} \\ & 0 & \end{array}$	$(3, 6)$	$g_1(p, p^2)\, g_1(p^3, p^4)\, g_2(p^4, p^3)$	$\lvert p\rvert^5 \phi(p^3)$
$\begin{array}{ccc} 6 & \boxed{3} & 0 \\ & 0 & \end{array}$	$(3, 6)$	$g_1(p^7, p^6)\, g_2(p^2, p^3)$	$-\lvert p\rvert^2 \phi(p^6)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & 0 \\ & 0 & \end{array}$	$(3, 8)$	$g_1(p^7, p^8)\, g_2(p^2, p^3)$	$-g_2(1, p)\lvert p\rvert^9$

According to the above table, we have

$$F^{(p)}(s_2;p^3) = -|p|^2\Psi(p^3,1) + \frac{|p|g_2(1,p)\phi(p^3)\Psi(p^3,p^2)}{|p|^{4s_2}} + \frac{|p|^4 g_2(1,p)\phi(p^3)\Psi(p^3,p^5)}{|p|^{10s_2}} + \frac{|p|^5\phi(p^3)\Psi(p^3,p^6)}{|p|^{12s_2}} - \frac{|p|^2\phi(p^6)\Psi(p^3,p^6)}{|p|^{12s_2}} - \frac{g_2(1,p)|p|^9\Psi(p^3,p^8)}{|p|^{16s_2}}. \tag{44}$$

By adding and subtracting certain necessary terms at vertices $(3,3)$ and $(3,5)$, and using the fact that $g_1(1,p)g_2(1,p) = |p|$ when $n = 3$, we find that $F^{(p)}(s_2;p^3)$ equals

$$\begin{aligned} &- |p|^2\Psi(p^3,1)\left[1 + \frac{\phi(p^3)}{|p|^{6s_2}}\frac{\Psi(p^3,p^3)}{\Psi(p^3,1)} + \frac{\phi(p^6)}{|p|^{12s_2}}\frac{\Psi(p^3,p^6)}{\Psi(p^3,1)} + \frac{g_2(1,p)|p|^7}{|p|^{16s_2}}\frac{\Psi(p^3,p^8)}{\Psi(p^3,1)}\right] \\ &+ \frac{g_2(1,p)|p|\,\phi(p^3)\Psi(p^3,p^2)}{|p|^{4s_2}}\left[1 + \frac{\phi(p^3)}{|p|^{6s_2}}\frac{\Psi(p^3,p^5)}{\Psi(p^3,p^2)} + \frac{g_1(1,p)p^3}{|p|^{8s_2}}\frac{\Psi(p^3,p^6)}{\Psi(p^3,p^2)}\right] \\ &+ \frac{|p|^2\phi(p^3)\Psi(p^3,p^3)}{|p|^{6s_2}}\left[1 + \frac{|p|g_2(1,p)}{|p|^{4s_2}}\frac{\Psi(p^3,p^5)}{\Psi(p^3,p^3)}\right]. \end{aligned} \tag{45}$$

After analyzing the terms in the bracketed sums, ignoring complications from the function Ψ as before, we have

$$F^{(p)}(s_2;p^3) = -|p|^2\Psi(p^3,1)\mathcal{D}_1^{(p)}(s_2,\Psi',p^7) + \frac{g_2(1,p)|p|\phi(p^3)\Psi(p^3,p^2)}{|p|^{4s_2}}\mathcal{D}_1^{(p)}(s_2,\Psi'',p^3) + \frac{|p|^2\phi(p^3)\Psi(p^3,p^3)}{|p|^{6s_2}}\mathcal{D}_1^{(p)}(s_2,\Psi''',p). \tag{46}$$

Arguing similarly to the previous section, one can use these local contributions to reconstruct the global Dirichlet series via twisted multiplicativity. The resulting objects satisfy the global functional equation for σ_2 as in (36).

3.2.3 The functional equation σ_2 with $k_1 = 6$.

As a final example, the set of all $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ with $k_1 = 6$ involves 7 support vertices and 18 BZL-patterns. In the case $n = 3$, however, only four of the BZL-patterns have non-zero Gauss sum products associated to them. These are listed in the table below.

Δ	$k(\Delta)$	$G(\Delta)$	$G(\Delta)$ for $n=3$
$\begin{array}{ccc} 6 & \boxed{3} & 0 \\ & \boxed{3} & \end{array}$	$(6,6)$	$g_1(p^7,p^6)\, g_2(p^2,p^3)\, g_2(p^2,p^3)$	$\lvert p\rvert^4\, \phi(p^6)$
$\begin{array}{ccc} \boxed{4} & 3 & \boxed{2} \\ & \boxed{3} & \end{array}$	$(6,6)$	$g_1(p^1,p^2)\, g_1(p^3,p^4)\, g_2(p^2,p^3)\, g_2(p^4,p^3)$	$-\lvert p\rvert^7\, \phi(p^3)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & 0 \\ & 3 & \end{array}$	$(6,8)$	$g_1(p^7,p^8)\, g_2(p^2,p^3)\, g_2(p^4,p^3)$	$-\lvert p\rvert^9 g_2(1,p)\phi(p^3)$
$\begin{array}{ccc} 6 & \boxed{5} & \boxed{2} \\ & \boxed{1} & \end{array}$	$(6,8)$	$g_1(p^1,p^2)\, g_1(p^7,p^6)\, g_2(1,p)\, g_2(p^4,p^5)$	$\lvert p\rvert^{11} g_2(1,p)\phi(p^6)$

Upon first inspection, it is unclear how to package the Gauss sum products neatly into p parts of Kubota Dirichlet series, as in the previous examples. However, the two nonzero terms at $(6,6)$ cancel each other out when $n=3$, as do the two nonzero terms at $(6,8)$. This seems like a very complicated way to write 0, but we remind the reader that the definition in terms of Gauss sums is "uniform" in n, in the sense that only the order of the multiplicative character in the Gauss sum changes. For other n, the p-part $H^{(n)}(p^{k_1},p^{k_2};p^2,p)$ with $k_1=6$ will have the same 18 products of Gauss sums, four of which are as shown in the third column of the table above. However, the evaluations as in the last column of the table depend on the choice of n and result in a different organization as Kubota Dirichlet series.

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