A Matrix Form of the Jacobi-Trudi Identity

Neranga Fernando
Advisor - Dr. Xiang-Dong Hou

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Definition

Let $\lambda_1 \geq 0$,. . . , $\lambda_n \geq 0$ be integers.

$$S_{\lambda_1,\ldots,\lambda_n}(x_1,\ldots,x_n) = \frac{det[x_i^{\lambda_j+n-j}]_{1\leq i,j\leq n}}{det[x_i^{n-j}]_{1\leq i,j\leq n}}$$

$S_{\lambda_1,\ldots,\lambda_n}(x_1,\ldots,x_n)$ is a symmetric polynomial in $x_1,\ldots,x_n$.

Reason :

- $det[x_i^{n-j}]_{1\leq i,j\leq n} \mid det[x_i^{\lambda_j+n-j}]_{1\leq i,j\leq n}$
- *both* det $[x_i^{n-j}]$ and $det[x_i^{\lambda_j+n-j}]$ are antisymmetric

Significance of the Schur Function

Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$; $(\lambda_1 \geq \lambda_2 \geq \ldots \lambda_m \geq 0, \lambda_1 + \ldots \lambda_m =$ n)

Let W^λ be the Weyl module of $GL(m, \mathbb{C})$.

($W^\lambda = h(\Sigma_\lambda) V^{\bigotimes n}$ where $V = \mathbb{C}^m$, $\sum_\lambda$ is the canonical SYT of shape λ, h(Σ_λ) is the Young symmetrizer of Σ_λ.)

Let

ϕ_λ be the character of W^λ . $\forall A \,\epsilon$ GL(m, C) with eigenvalue $x_1, \ldots, x_m$.

ϕ_λ (A) $= S_\lambda(x_1, \ldots, x_m)$ (Weyl's character formula)

Complete/Elementary Symmetric Functions

n^{th} Elementary Symmetric Function

$$e_n(x_1, \ldots, x_n) = \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n}$$

n^{th} Complete Symmetric Function

$$h_n(x_1, \ldots, x_n) = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n}$$

Note : If $f(x_1, x_2, \ldots)$ is a symmetric function define

$$f(x_1, x_2, \ldots) = f\,(x_1, \ldots, x_n, 0, 0, \ldots)$$

Example

$$h_3(x_1, x_2) = x_1^3 + x_2^3 + x_1 x_2^2 + x_1^2 x_2$$

The Jacobi-Trudi Identity

Theorem A

Let $\lambda_1 \geq 0$,. . . , $\lambda_n \geq 0$ be integers. Then

$$det[h_{\lambda_i - i + j}]_{1 \leq i,j \leq n} = \frac{det[x_i^{\lambda_j + n - j}]_{1 \leq i,j \leq n}}{det[x_i^{n-j}]_{1 \leq i,j \leq n}}$$

where h_r is defined to be 0 if $r < 0$.

(Note - The right side of the above equation is the **Schur Function** $S_\lambda(x_1,. . . ,x_n)$ where $\lambda = (\lambda_1,. . . ,\lambda_n))$

Theorem B

Let $\mathbb{F} = \mathbb{Q}\,(x_1, \ldots, x_n)$ with $n \geq 1$ and let s ≥ 0

Then in $\mathbb{F}$ we have

$$\textbf{(0.1)} \quad \sum_{k=1}^{n} \frac{x_k^s}{\prod_{\substack{1 \leq i \leq n \\ i \neq k}} (x_k - x_i)} = h_{s-n+1}(x_1, \ldots, x_n)$$

where $h_r(x_1, \ldots.., x_n)$ is the r^{th} complete symmetric function in $x_1, \ldots, x_n$.

(1) Assume s $\geq$ 0.

Let a_s denote the left side of (0.1).

Define

$$P(t) = \sum_{k=1}^{n} \frac{x_k^s}{\prod_{\substack{1\leq i\leq n \\ i\neq k}}(x_k - x_i)} \prod_{\substack{1\leq i\leq n \\ i\neq k}}(t - x_i) \in \ \mathbb{F}[t]$$

$\mathrm{P(t)} = a_s t^{n-1} +$ lower terms of t

$\mathrm{P}(\mathrm{x}_k) = x_k^s \ , \quad 1\leq k \leq n$

(0.2) $\mathrm{t}^s - P(t) \ \equiv \ 0 \quad (mod \ \prod_{1\leq k\leq n}(t - x_k))$

Proof of Theorem B Continues

Case (1) Assume 0 $\leq$ s $\leq$ n-1.

Since $deg(t^s - P(t)) < n$.

Eq. (0.2) gives P(t)= t^s

$$a_s = \begin{cases} 1 & \text{if} \quad s = n-1 \\ 0 & \text{if} \quad 0 \leq s < n-1 \end{cases}$$

which is consistent with (0.1).

Case (2) Assume s $\geq$ n.

By (0.2),

there exists a monic polynomial $g(x) \epsilon \mathbb{F}[t]$ of degree s-n such that

$$t^s - P(t) = g(t) \prod_{1 \leq k \leq n} (t - x_k)$$

Proof of Theorem B Continues

So

$t^{s-n}g(\frac{1}{t})\sum_{i=0}^{n}(-1)^i e_i t^i = 1 - t^s P(\frac{1}{t})$

where e_i is the i^{th} elementary symmetric polynomial in $x_1, \ldots, x_n$.

Let

$g_1(t) = t^{s-n}g(\frac{1}{t})$. Then deg $g_1 \leq s - n$

(0.3) $g_1(t)\sum_{i=0}^{n}(-1)^i e_i t^i \equiv$ 1 - $a_s t^{s-n+1}$ (mod t^{s-n+2})

In particular,

(0.4) $g_1(t)\sum_{i=0}^{n}(-1)^i e_i t^i \equiv$ 1 (mod t^{s-n+1})

Note that g_1 is uniquely determined by (0.4) and the condition $deg\, g_1 \leq s - n$.

Proof of Theorem B Continues

It is well known that

$$(\sum_{j=0}^{\infty} h_j t^j)(\sum_{i=0}^{\infty} e_i(-t)^i) \ = 1$$

Note that $e_i = 0$ when $i > n$

$$(\sum_{j=0}^{s-n} h_j t^j)(\sum_{i=0}^{n} e_i(-t)^i) \ \equiv 1 \ (\text{mod } \mathrm{t}^{s-n+1})$$

Therefore $\mathrm{g}_1(t) = \sum_{j=0}^{s-n} h_j t^j$.

(0.5)

$$\mathrm{g}_1(t) \sum_{i=0}^{n} e_i(-t)^i \ = \ 1 - (\sum_{j=s-n+1}^{\infty} h_j t^j)(\sum_{i=0}^{n} e_i(-t)^i)$$

$$\mathrm{g}_1(t) \sum_{i=0}^{n} e_i(-t)^i \ \equiv \ 1 - \mathrm{h}_{s-n+1} t^{s-n+1} \ (\text{mod } \mathrm{t}^{s-n+2})$$

comparing (0.5) and (0.3), we have $\mathrm{a}_s = h_{s-n+1}$

Theorem C - Main Theorem

Let $\lambda_1 \geq 0$,. . . , $\lambda_n \geq 0$ be integers.

$$[\mathrm{h}_{\lambda_i - i + j}]_{1 \leq i,j \leq n} = [x_k^{\lambda_i + n - i}]_{1 \leq i,k \leq n} \cdot [x_k^{n-i}]^{-1}_{1 \leq i,k \leq n} \cdot [h_{-i+j}]_{1 \leq i,j \leq n}$$

- Then **Jacobi-Trudi Identity** is a Corollary of Theorem C , Since $[h_{-i+j}]_{1 \leq i,j \leq n}$ is uni upper triangular.

Proof of Theorem C

$$[\mathrm{h}_{\lambda_i - i + j}]_{1 \le i,j \le n}$$

$$= [\ \sum_{k=1}^{n} \frac{x_k^{\lambda_i - i + j + n - 1}}{\prod_{\substack{1 \le l \le n \\ l \ne k}} (x_k - x_l)}]_{1 \le i,j \le n} \quad (By Theorem\ B)$$

$$= [\ \sum_{k=1}^{n} x_k^{\lambda_i + n - i} \frac{x_k^{j-1}}{\prod_{\substack{1 \le l \le n \\ l \ne k}} (x_k - x_l)}]_{1 \le i,j \le n}$$

$$= [\mathrm{x}_k^{\lambda_i + n - i}]_{1 \le i,k \le n} \cdot [\frac{x_k^{j-1}}{\prod_{\substack{1 \le l \le n \\ l \ne k}} (x_k - x_l)}]_{1 \le k,j \le n}$$

$$= [\mathrm{x}_k^{\lambda_i + n - i}]_{1 \le i,k \le n} \cdot [x_k^{n-i}]_{1 \le i,k \le n}^{-1} \cdot [x_k^{n-i}]_{1 \le i,k \le n} \cdot [\frac{x_k^{j-1}}{\prod_{\substack{1 \le l \le n \\ l \ne k}} (x_k - x_l)}]_{1 \le k,j \le n}$$

Proof of Theorem C Continues

$$= [x_k^{\lambda_i+n-i}]_{1\leq i,k\leq n} \cdot [x_k^{n-i}]^{-1}_{1\leq i,k\leq n} \cdot [\sum_{k=1}^{n} \frac{x_k^{n-i+j-1}}{\prod_{\substack{1\leq l\leq n \\ l\neq k}}(x_k - x_l)}]_{1\leq i,j\leq n}$$

$$= [\ x_k^{\lambda_i+n-i}]_{1\leq i,k\leq n} \cdot [x_k^{n-i}]^{-1}_{1\leq i,k\leq n} \cdot [h_{-i+j}]_{1\leq i,j\leq n}$$

(By Theoerm B)

References

- I.G. Macdonald, *Symmetric Functions and Orthogonal Polynomials*, University Lecture Series, 12 American Mathematical Society, Providence, RI, 1998.
- Online Resources