Quandles, racks and shelves

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Introduction

Knots

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We usually think of knots in three-dimensional space.

Introduction (contd.)

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k_0 is *ambient isotopic* to K_1 if there is a continuous map

$$H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

such that $H(K_0, 0) = K_0$, $H(K_0, 1) = K_1$ and $H(x, t)$ is injective for every $t \in [0,1]$.

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Such a map is called an *ambient isotopy*.

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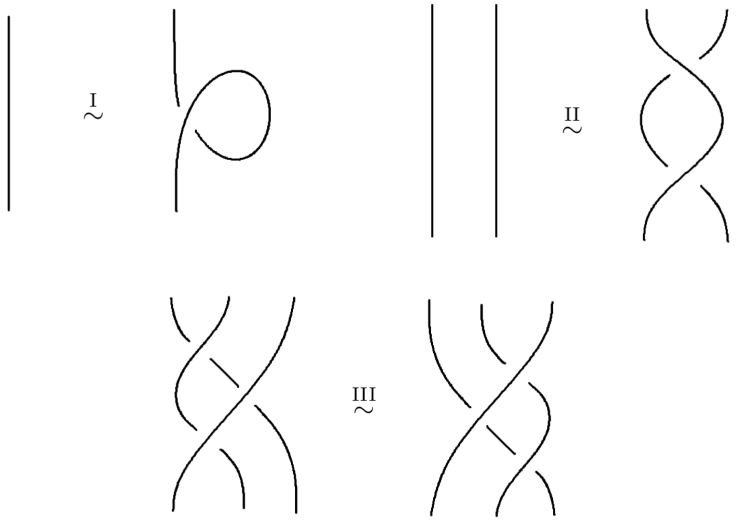
Two knot diagrams, K_0 and K_1, represent the same knot type, if we can identify an explicit sequence of three moves taking K_0 to K_1, and these moves are now called **Reidemeister moves**.

Reidemeister moves



Knot Invariants

A **knot invariant** is a function $f : \mathcal{K} \to X$ from the set of all knot diagrams to a set X such that for each Reidemeister move, we have

$$f(K_1) = f(K_2)$$

where K_1 is the knot diagram before the move and K_2 is the same diagram after the move.

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If f is a knot invariant, then any two diagrams related by Reidemeister moves must give the same value when we evaluate f.

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A **tricoloring** of a knot diagram is a choice of color for each arc in the diagram from a set of three colors.

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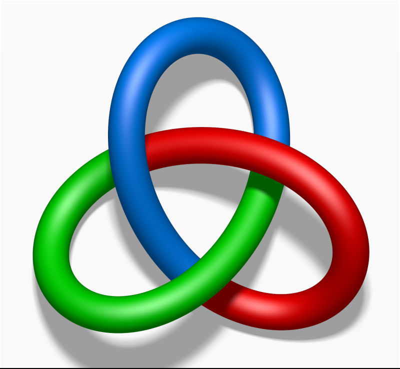


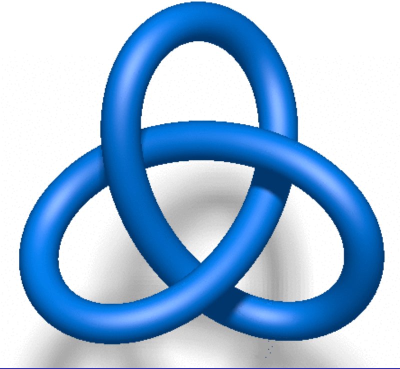
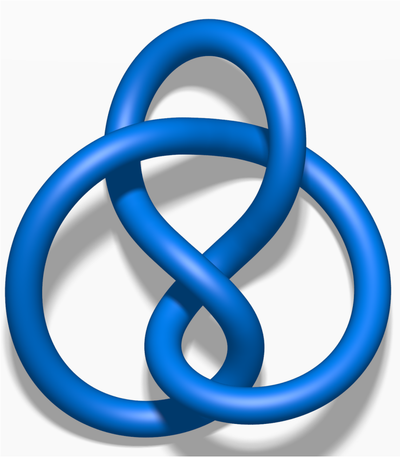


Figure-eight knot



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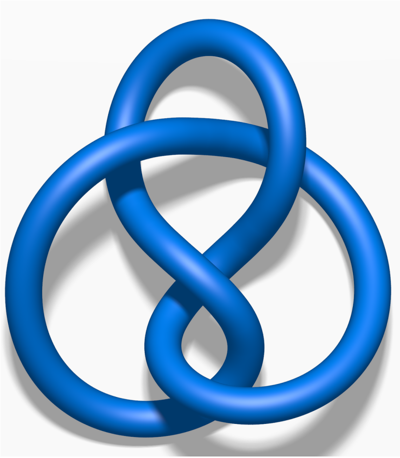

Not tri-colorable

Kei

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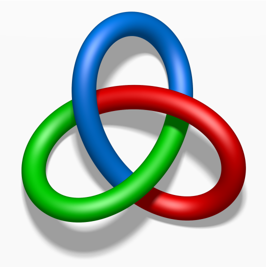
In general, the Kei operation is *nonassociative*.

Connection of the three axioms to Knots



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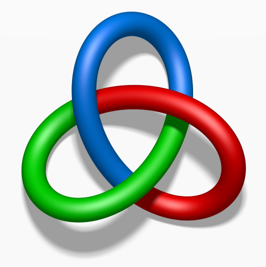

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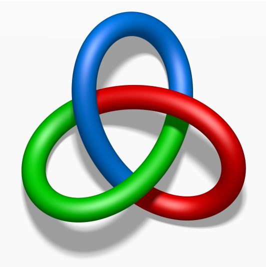

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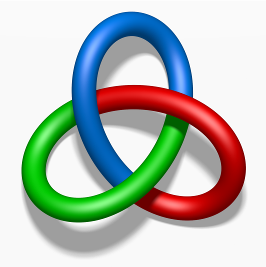

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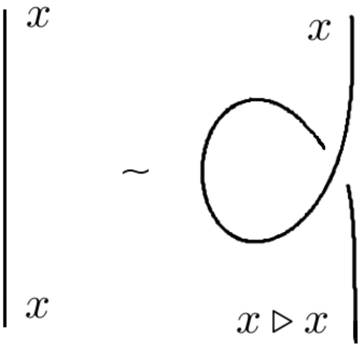

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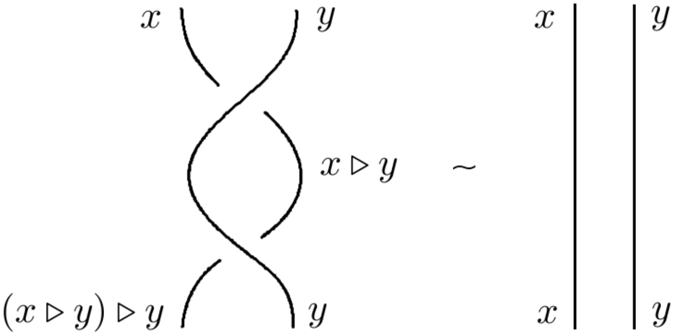
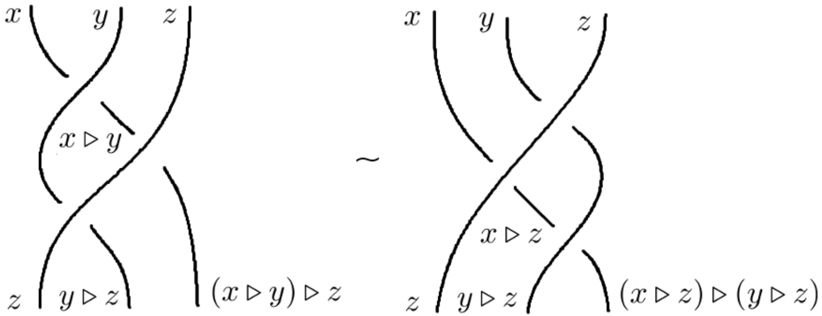
Kei axiom (ii) follows from the type II Reidemeister move:

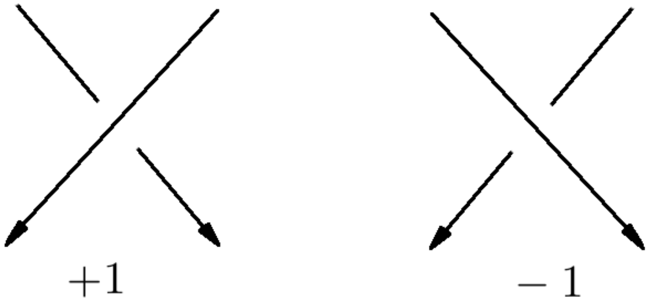


Connection of the three axioms to Knots (contd.)

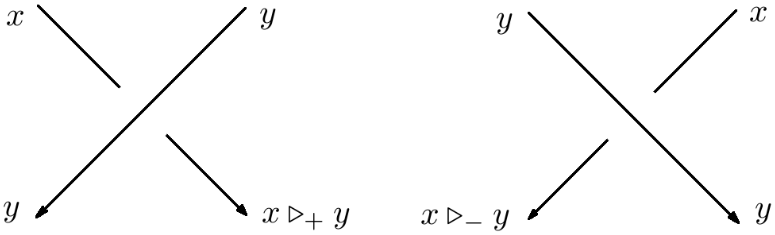
Kei axiom (iii) follows from the type III Reidemeister move:



Including a choice of orientation for knots



Including a choice of orientation for knots (contd.)



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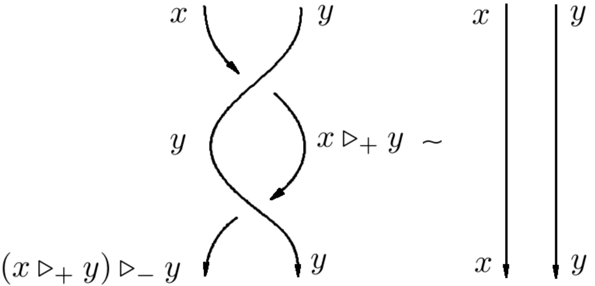

Including a choice of orientation for knots (contd.)

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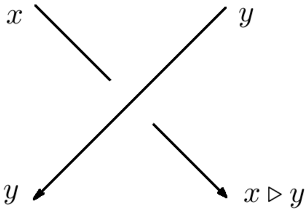

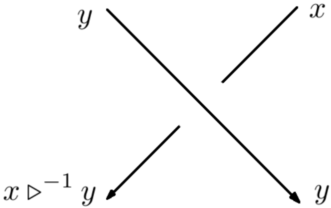

Quandles

A *quandle* is a set X with a binary operation $\triangleright : X \times X \to X$ satisfying:

(i) For all $x \in X$, $x \triangleright x = x$

(ii) For all $y \in X$, the map $\beta_y : X \to X$ defined by $\beta_y(x) = x \triangleright y$ is invertible.

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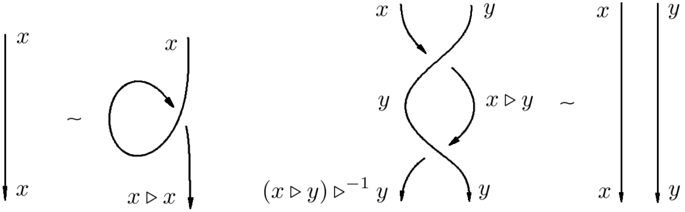
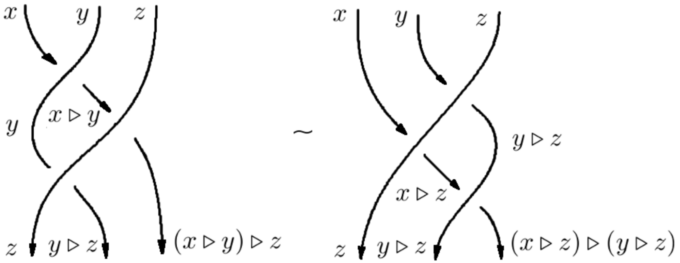
David Joyce and S.V. Matveev studied these structures independently in the 1980s.

Quandle Axioms and Reidemeister Moves



Quandle Axioms and Reidemeister Moves (contd.)



Examples of Quandles

Let $X = \mathbb{Z}$ or $\mathbb{Z}_n$ and define

$$x \triangleright y = 2y - x.$$

Axiom 1 $x \triangleright x = 2x - x = x$

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$(x \triangleright z) \triangleright (y \triangleright z) = 2(y \triangleright z) - (x \triangleright z) = 2(2z - y) - (2z - x) = 2z - 2y + x$

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This is called the **Dihedral Quandle**.

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$\mathrm{orb}(0) = \mathrm{orb}(2) = \{0, 2\}$ and $\mathrm{orb}(1) = \mathrm{orb}(3) = \{1, 3\}$

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Example 4 More generally, let G be any group. Then G is a quandle under the operation of conjugation, i.e.

$$x \triangleright y = y^{-1}xy$$

Examples of Quandles

Let A be a module over $\Lambda = \mathbb{Z}[t^{\pm}]$. Then A is a quandle under the operation

$$\vec{x} \triangleright \vec{y} = t\vec{x} + (1-t)\vec{y}$$

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Example 5 Any vector space V becomes an Alexander quandle when we select an invertible linear transformation $t : V \to V$ and define

$$\vec{x} \triangleright \vec{y} = t\vec{x} + (I - t)\vec{y}$$

where I is the identity matrix.

Examples of Quandles

Consider $V = \mathbb{R}^2$ and choose $t = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Then t is invertible with $t^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ and $I - t = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix}$;

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleright \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleright \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - y_2 \\ x_1 + 3x_2 - y_1 - 2y_2 \end{bmatrix}$$

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$x \triangleright y = x$

$\triangleright$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$x \triangleright y = 2x + 2y$

$\triangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

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For example,

$$(1 + t) \triangleright 2t = t(1 + t) + (1 - t)(2t) = 1 + t$$

Racks

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A *rack* is a set X with a binary operation $\triangleright : X \times X \to X$ satisfying:

(i) For all $y \in X$, the map $\beta_y : X \to X$ defined by $\beta_y(x) = x \triangleright y$ is invertible.

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The set of Quandles $\subset$ The set of Racks

Quandle Rings

V.G. Bardakov, I.B.S. Passi, and M. Singh, **Quandle rings**, J. Algebra Appl. (2019)

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Let X be a rack and R an associative ring with unity. They introduced the rack ring $R[X]$, a nonassociative ring, of X with coefficients in R.

$$R[X] := \left\{ \sum_i \alpha_i x_i \,|\, \alpha_i \in R,\ x_i \in X \right\}$$

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Mohamed Elhamdadi, Boris Tsvelikhovskiy, N.F., **Ring theoretic aspects of quandles**. Journal of Algebra (2019)

Shelves

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The set of Quandles $\subset$ The set of Racks $\subset$ The set of Shelves

Matthew Pradeep Goonewardena, Mohamed Elhamdadi and N.F., **Isomorphism classes of shelves** (soon to be submitted)

Example of a Shelf

Let $X = \mathbb{Z}_4$ and $x \triangleright y = x + 2y$ for all $x, y \in X$. Then $(\mathbb{Z}_4, *)$ is a shelf of order 4.

$\triangleright$	0	1	2	3
0	0	1	2	3
1	2	3	1	3
2	0	0	2	0
3	2	1	3	1

Number of Connected Racks and Quandles

n	number of connected racks	number of connected quandles
1	1	1
2	1	0
3	2	1
4	2	1
5	4	3

Thank you!