A Study of Knots and Quandles

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Introduction

A *knot* is a simple closed curve, where “simple” means it doesn’t intersect itself and “closed” means it has no loose ends. For example, consider an extension cord with two loose ends plugged in. The purpose of *knot theory* is to have a systematic way to distinguish two given knots.

Knot diagrams and orientation of knots

A *knot diagram* is a two-dimensional projection of a three-dimensional knot.

Example 1 Here are knot diagrams of trefoil knot and figure-8 knot:

Figure 1. The Knot Dia- gram of Trefoil Knot.

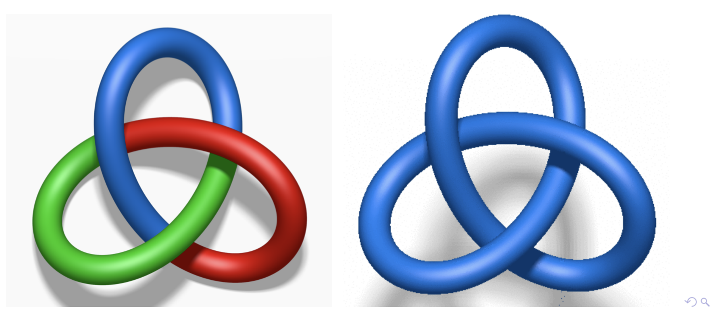
Figure 2. The Knot Dia- gram of Figure-Eight Knot.

Example 2 At every crossing, we have *positive crossing* or *negative crossing*.

Purpose of knot theory and knot invariants

The purpose of knot theory is to understand the properties and classification of knots. This involves determining whether two knots are equivalent and finding ways to distinguish between different knots.
A *knot invariant* is a quantity or property that remains unchanged under ambient isotopy. Knot invariants are used to distinguish between different knots, as equivalent knots have the same invariants.

Example 3 Fox tricoloring, which is a knot invariant, was introduced by Ralph Fox in the 1950s. A tricoloring is valid if at every crossing we either have all three colors the same or all three colors different. A valid tricoloring is nontrivial if it uses all three colors. The trifoil knot below has valid tricoloring which is nontrivial.



Quandle and Reidemeister moves

Let X be a non-empty set, and let $\triangleright : X \times X \mapsto X$ be a binary operation. The pair $(X, \triangleright)$ is called a quandle if it satisfies the following axioms:
1. For all $x \in X$, $x \triangleright x = x$.
2. For all $x, y \in X, \beta_y(x) = x \triangleright y$ is invertible.
3. For all $x, y, z \in X, (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Example 4 Let $X = \mathbb{F}_3$. For $x, y \in X$, define $x \triangleright y = 2y - x \pmod 3$.

$\triangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

This is the dihedral quandle which also called the Takasaki Kei.

Example 5 Here are the three Reidemeister moves:

Alexander quandle

Let A be a module over $\Lambda = \mathbb{Z}[t^{\pm}]$. Then A is a quandle under the operation $\vec{x} \triangleright \vec{y} = t\vec{x} + (1-t)\vec{y}$. This is known as an Alexander quandle.

Example 6 Any vector space V becomes an Alexander quandle when we select an invertible linear transformation $t : V \to V$ and define $\vec{x} \triangleright \vec{y} = t\vec{x} + (I - t)\vec{y}$ where I is the identity matrix.
Consider $V = \mathbb{R}^2$ and choose $t = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then t is invertible and $I - t = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix}$. Then we have quandle operation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleright \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - y_2 \\ x_1 + 3x_2 - y_1 - 2y_2 \end{bmatrix}$$

The integers modulo n, $\mathbb{Z}_n$, form an Alexander quandle with the choice of any invertible element $t \in \mathbb{Z}_n$. Consider $\mathbb{Z}_3$. We can choose $t = 1$ or $t = 2$. Then we have Alexander quandle operation

$$\vec{x} \triangleright \vec{y} = t\vec{x} + (1-t)\vec{y}$$

$x \triangleright y = x$

$\triangleright$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$x \triangleright y = 2x + 2y$

$\triangleright$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Alexander polynomial

The *Alexander polynomial* is an invariant of a knot. It is a Laurent polynomial so it can have t with negative powers. The way to compute the Alexander polynomial is to take the determinant of an $(n-1) \times (n-1)$ minor of the presentation matrix A.

Example 7 Let’s compute the Alexander polynomial of the trefoil knot here:
We have the presentation matrix $A = \begin{bmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{bmatrix}$. The Alexander polynomial is the determinant of any 2×2 minor of A. Let’s pick $\begin{bmatrix} 1-t & -1 \\ t & 1-t \end{bmatrix}$. The determinant of the minor is $-t^{-2} + 1 + t$. We can multiply through by $-t^2$ to get normalized polynomial $1 - t^2 - t^3$.

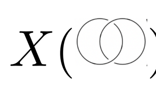
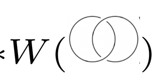
Jones polynomial

The *Jones polynomial* is a knot invariant. It is generated through the *bracket polynomial*.

Example 8 Here is how we compute the bracket polynomial and jones polynomial of the hopf link.

$$= A(-A^3) + A^{-1}(-A^{-3}) = -A^4 - A^{-4}$$

This is the bracket polynomial of the hopf link. Now we have to calculate the X polynomial of hopf link.

$X($$) = (-A)^{-3*W(}$$^{)}(-A^4 - A^{-4}) = -A^{-2} - A^{-10}$

Then, we replace all A with $t^{-1/4}$ and get the jones polynomial of the hopf link

$$= -t^{1/2} - t^{5/2}$$

Future plans

We plan on studying the interplay between quandles and reversed Dickson permutation polynomials. We also plan on exploring the application of knot theory in fields such as biology (DNA knotting), chemistry (molecular knotting), and physics (the study of quantum field theories).

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