(1) Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

  (a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.
  (b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."
  (c) Every nonempty open set contains a rational number.
  (d) Every bounded infinite closed set contains a rational number.

(2) Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.
  (a) $\overline{A \cup B}$
  (b) $A \setminus B = \{x \in A : x \notin B\}$
  (c) $(A^c \cup B)^c$
  (d) $(A \cap B) \cup (A^c \cap B)$
  (e) $\overline{A}^c \cap \overline{A^c}$

(3) Let A be nonempty and bounded above so that $s = \sup A$ exists.

  (a) Show that $s \in \overline{A}$
  (b) Can an open set contain its supremum?

(4) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\cap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbb{R}$.

(5) Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\emptyset$.

(6) Use the ϵ-δ definition to supply a proper proof for the following limit statements.

  (a) $\lim_{x\to 2} 3x + 1 = 7$.
  (b) $\lim_{x\to 3} 5x - 6 = 9$.
  (c) $\lim_{x\to 2} 3x + 4 = 10$.
  (d) $\lim_{x\to 3} 1/x = 1/3$.
  (e) $\lim_{x\to 0} x^3 = 0$.

(7) For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.

  (a) $\lim_{x\to 4} \sqrt{x} = 2$, where $\epsilon = 1$.
  (b) $\lim_{x\to 3} 5x - 6 = 9$, where $\epsilon = 1$.

(8) Decide if the following claims are true or false, and give short justifications for each conclusion.

  (a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.
  (b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.
  (c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x) - 2]^2 = 3(L - 2)^2$.
  (d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

(9) Let $g : A \to R$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

(10) **Definition** $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.
  (a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition. .
  (b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} 1/x = 0$
  (c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ look like? Give an example of such a limit.