(1) Let (a_n) be a bounded sequence.

  (a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

  (b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by
$$\limsup a_n = \lim y_n,$$
where y_n is the sequence from part (a). Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

  (c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

  (d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

(2) Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

(3) Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

  (a) $c_n = |a_n - b_n|$

  (b) $c_n = (-1)^n a_n$

  (c) $c_n = \lfloor a_n \rfloor$

(4) Find examples of two series $\sum a_n$ and $\sum a_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

(5) (a) Show that if $a_n > 0$ and $\lim(na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

  (b) Assume that $a_n > 0$ and $\lim(n^2 a_n)$ exists. Show that $\sum a_n$ converges.

(6) Decide whether each of the following series converges or diverges:

  (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

  (b) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

  (c) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(7) Give an example of each or explain why the request is impossible referencing the proper theorem(s).

  (a) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

  (b) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

(8) Use the subsequences (s_{2n}) and (s_{2n+1}), and the Monotone Convergence Theorem to prove the Alternating Series Test.

(9) Give a proof for the Comparison Test using the Monotone Convergence Theorem.

(10) Prove the following:
  (a) The union of an arbitrary collection of open sets is open.
  (b) The intersection of a finite collection of open sets is open.
  (c) The union of a finite collection of closed sets is closed.
  (d) The intersection of an arbitrary collection of closed sets is closed.