(1) Give an example of each of the following, or show that no such example is possible.

(a) A sequence with no convergent subsequences.
(b) A sequence (a_n) such that (a_{3k}) converges to 1, (a_{3k+1}) converges to 3 and (a_{3k+2}) converges to 6.
(c) A sequence (x_n) such that the subsequence x_{3k} converges to 3 and the subsequence x_{7k} converges to 7.
(d) A sequence that has subsequences that converge to every natural number n.

(2) Use subsequences to prove that the sequence $x_n = \frac{n(-1)^n}{n+1}$ diverges.

(3) Prove that if the subsequences (a_{2k}) and (a_{2k+1}) both converge to the same limit a, then the sequence (a_n) converges to a.

(4) Use the result of Problem #3 to prove that the sequence $x_n = \frac{n+(-1)^n}{n+1}$ converges, and find its limit.

(5) Let $z_1 = \sqrt{2}$ and $z_{n+1} = 6 - 2|z_n - 3|$. Prove that (z_n) has a subsequence that converges.

(6) Let $a_1 = 5$ and $a_{n+1} = -\frac{1}{2}a_n + 4$. Prove (a_n) converges and find its limit.

(7) Let $x_1 = 1$ and $x_{n+1} = \pi \sin(x_n)$ for $n \geq 1$. Prove (x_n) has a convergent subsequence.

(8) Let $z_1 = 1$ and $z_{n+1} = \frac{2}{3}z_n(6 - z_n)$. Prove that (z_n) has a subsequence that converges.

(9) Prove that every sequence has a monotone subsequence.

(10) Prove that there exists a sequence that has subsequences that converge to every real number.

(11) Define $a_1 = 1$ and $a_{n+1} = \sqrt{5 - 2a_n}$. Show that the sequence (a_n) is contractive.

(12) Let (x_n) be a sequence of real numbers defined by

$$x_1 = 1, x_2 = 2, x_{n+2} = \frac{1}{3}(2x_{n+1} + x_n), \text{ for all } n \in \mathbb{N}.$$

Show that (x_n) is convergent and find its limit.

(13) If $x_1 > 0$ and $x_{n+1} = (2 + x_n)^{-1}$ for $n \geq 1$, show that the sequence (x_n) is contractive.