We know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

(1) Prove that

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n \quad \text{for } |x| < \frac{1}{2}.$$

(2) Find a power series expansion centered at 0 for

$$f(x) = \frac{1}{2+x^2}$$

and find the interval of convergence.

Theorem Assume that

$$F(x) = \sum_{n=0}^{\infty} c_n \, (x-a)^n$$

has radius of convergence $R > 0$. Then $F(x)$ is differentiable on $(c - R, c + R)$ (or for all x if $R = \infty$). Furthermore, we can integrate and differentiate term by term for $x \in (c - R, c + R)$,

$$F'(x) = \sum_{n=1}^{\infty} n \, c_n \, (x-a)^{n-1}$$

$$\int F(x)\, dx \;=\; A + \sum_{n=0}^{\infty} \frac{c_n}{n+1} \, (x-a)^{n+1} \qquad (A \text{ any constant})$$

These series have the same radius of convergence R.

(3) Prove that for $-1 < x < 1$,

$$\frac{1}{(1-x)^2} \;=\; 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

(4) Prove that for $-1 < x < 1$,

$$\tan^{-1} x \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{2n+1}$$