(1) Prove the Squeeze Theorem using the ϵ-δ definition of a functional limit.

(2) Show using the ϵ-δ definition that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

(3) Let $g(x) = \sqrt[3]{x}$.

  (a) Prove that g is continuous at $c = 0$.

  (b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

(4) Provide an example of each or explain why the request is impossible.

  (a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0.

  (b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.

  (c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

  (d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.

  (e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

(5) (Composition of Continuous Functions) Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

  (a) Supply a proof of this theorem using the ϵ-δ definition of continuity.

  (b) Give another proof of this theorem using the sequential characterization of continuity

(6) Assume f and g are defined on all of $\mathbb{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$.

  (a) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r.$$

  (b) Show that the result in (a) does follow if we assume f and g are continuous.

  (c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

(7) (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

  (a) Show that f is continuous on $\mathbb{R}$.

  (b) Pick some point $y_1 \in \mathbb{R}$ and construct a sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

  In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

  (c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

  (d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in (b).

(8) Assume that $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that k is a closed set.

(9) Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

(10) Let $F \subseteq \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

(11) Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

(12) Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|].$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.

(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbb{N}$, define f_n on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), \ldots, f_n(x)\}$.