Normal edge-colorings of cubic graphs

Vahan Mkrtchyan

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Outline

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If $V = \{a, b, c, 1, 2\}$ and $E =< \{1,2\}, \{a,b\}, \{1,2\}, \{b,c\}, \{a,c\}, \{b,c\} >$, then $G = (V, E)$ is a graph.

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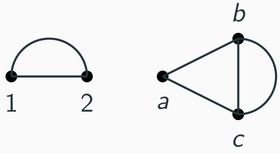

Figure 1: The drawing of $G = (V, E)$.

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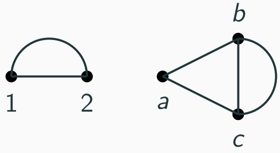

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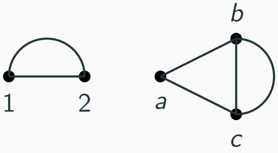

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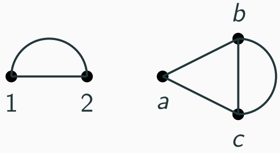

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Simple and parallel edges of G

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- A graph may contain parallel edges, but no loops.
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- The degree of a vertex is the number of edges incident to it.

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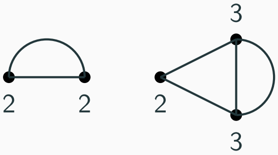

Figure 2: Degrees of vertices of $G = (V, E)$.

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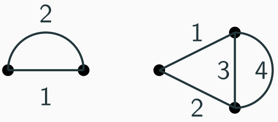

Figure 3: An example of a graph $G = (V, E)$ with $\chi'(G) = 4$.

Theorems of Shannon and Vizing

Maximum Degree and Multiplicity of a Graph

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Corollary (Vizing)

For any simple graph G (i.e. $\mu(G) = 1$): $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Some examples and results

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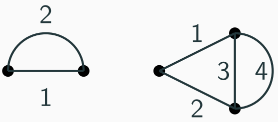
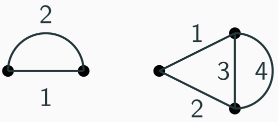


Figure 4: An example of a graph $G = (V, E)$ with $\Delta(G) = 3$, $\mu(G) = 2$ and $\chi'(G) = 4$.



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Corollary

If G is a cubic graph, then

$$3 \leq \chi'(G) \leq 4.$$

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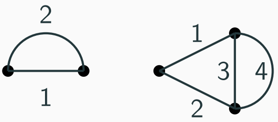

Figure 4: An example of a graph $G=(V,E)$ with $\Delta(G)=3$, $\mu(G)=2$ and $\chi'(G)=4$.

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If G is a cubic graph, then

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Theorem (Holyer)

The problem of deciding whether a given cubic graph G has $\chi'(G)=3$ is NP-complete.

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Figure 5: An example of a matching.

An example of a perfect matching in a graph

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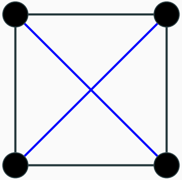

Figure 6: A perfect matching in K_4.

Bridges in graphs

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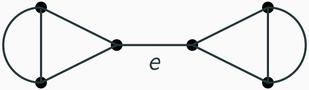

Figure 7: e is a bridge in a cubic graph G.

Cubic graphs and perfect matchings

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For a bridgeless cubic graph G, let $k(G)$ be the smallest number of perfect matchings covering the edge-set of G.

Observation

If G is a cubic graph, then

$$\chi'(G) = 3 \textit{ if and only if } k(G) = 3.$$

The Petersen graph P_{10} and its 6 perfect matchings

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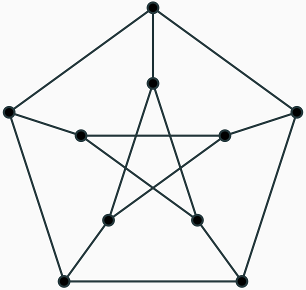

Figure 8: The graph P_{10}.

The Petersen graph P_{10} and its 6 perfect matchings

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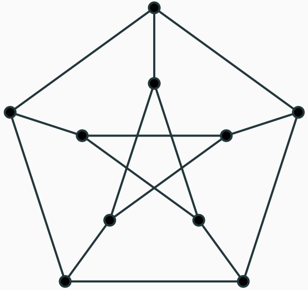

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The Petersen graph is the smallest bridgeless cubic graph G with $\chi'(G) = 4$.

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- $k(P_{10}) = 5$.

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Conjecture (Berge-Fulkerson restated)

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For any bridgeless cubic graph G we have

$$\chi'(2G) = 6.$$

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Observation (3-edge-colorable cubic graphs)

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Berge-Fulkerson conjecture holds for 3-edge-colorable cubic graphs G *(i.e.* $\chi'(G) = 3$*).*

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Theorem (G. Mazzuoccolo, 2011)

Conjectures of Berge and Berge-Fulkerson are equivalent.

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- $l(\mathscr{C})$ is called the length of the cycle cover $\mathscr{C}$.

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Any bridgeless (not necessarily cubic) graph G has a cycle cover $\mathscr{C} = (C_1, ..., C_t)$, such that $l(\mathscr{C}) \leq \frac{7}{5} \cdot |E|$.

The relationship among the three conjectures

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$(5, 2)$ *Even Subgraph Cover Conjecture implies Cycle Double Cover Conjecture.*

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Theorem (U. Jamshy, M. Tarsi, 1992)

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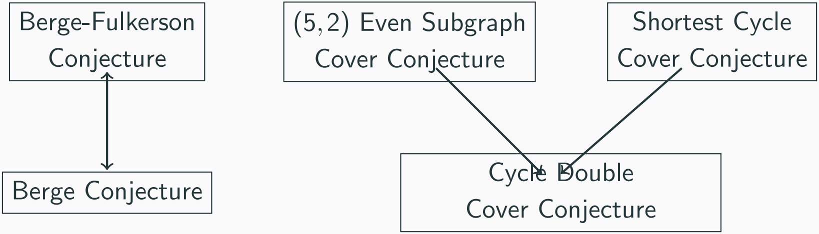

Figure 9: The relationship among the five conjectures.

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If G and H are two cubic graphs, then an H-coloring of G is a mapping $f : E(G) \to E(H)$ such that for each vertex $x \in V(G)$, there is a vertex $y \in V(H)$ with $f(\partial_G(x)) = \partial_H(y)$.

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If G admits an H-coloring f, then we will write $H \prec G$.

An example of an H-coloring

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An example: $H \prec G$

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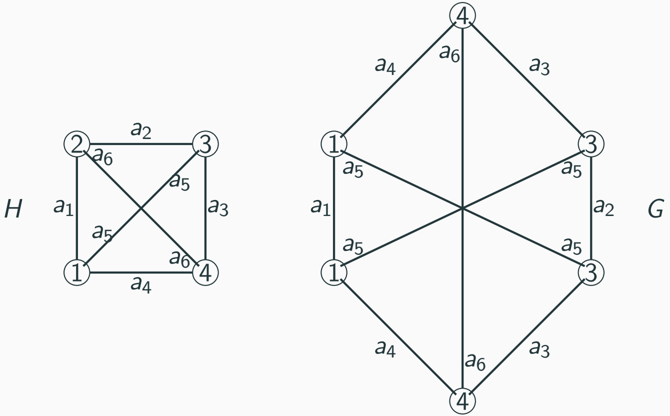

Figure 10: An example of an H-coloring of G.

Some properties

Some properties

Transitivity

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(f) *For every bridge e of G, the edge $f(e)$ is a bridge of H.*

(g) *If H is bridgeless, then G is also bridgeless.*

Petersen coloring conjecture

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Jaeger's unifying conjecture

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In 1988, Jaeger presented a conjecture, that has unified the conjectures about perfect matchings and cycle covers.

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Conjecture (P_{10}-conjecture, 1988)

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In 1988, Jaeger presented a conjecture, that has unified the conjectures about perfect matchings and cycle covers.

Conjecture (P_{10}-conjecture, 1988)

For any bridgeless cubic graph G, one has: $P_{10} \prec G$.

Consequences of P_{10}-conjecture

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Observation

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P_{10}-conjecture implies Berge-Fulkerson conjecture.

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P_{10}-conjecture implies Shortest Cycle Cover Conjecture.

The relationship among the six conjectures

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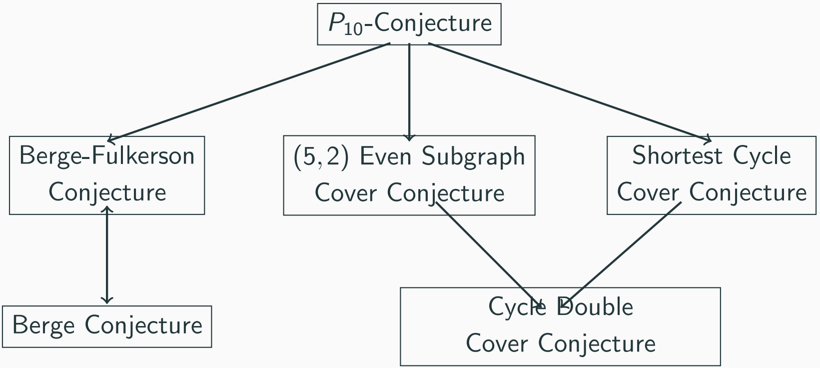

Figure 11: The relationship among the six conjectures.

Normal edge-colorings of cubic graphs

Snarks and their Petersen colorings

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A snark is a bridgeless cubic graph that is not 3-edge-colorable.

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If a cubic graph G admits a P_{10}-coloring f, such that for an edge e of P_{10} $f^{-1}(e) = \emptyset$,

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If a cubic graph G admits a P_{10}-coloring f, such that for an edge e of P_{10} $f^{-1}(e) = \emptyset$, then G is 3-edge-colorable.

Corollary

If G is a snark, then any P_{10}-coloring of G must use all edges of P_{10}.

Definition

Poor, rich edges and normal edge-colorings of cubic graphs

Definition

Let f be a k-edge-coloring of a cubic graph G, and let $S_f(w)$ be the set of colors of edges of G incident to the vertex w.

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A k-edge-coloring f of a cubic graph G is normal, if any edge of G is poor or rich in f. Let $\chi_N(G)$ be the smallest k for which G admits a normal k-edge-coloring.

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An example of a normal edge-coloring of a cubic graph

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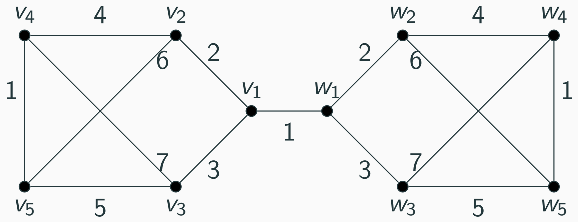

Figure 12: A cubic graph that requires 7 colors in a normal coloring.

An example of a normal edge-coloring of a cubic graph

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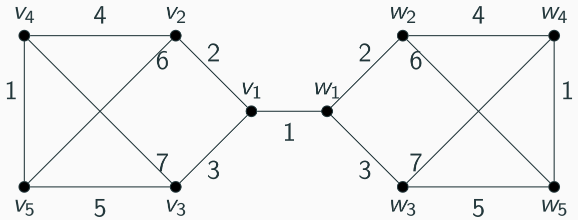

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The bridge is poor.

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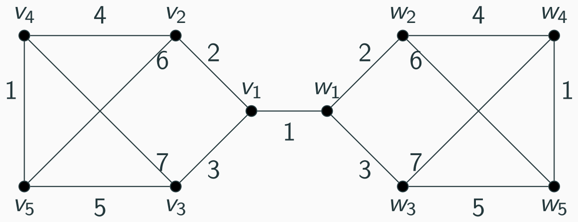

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The bridge is poor. All other edges are rich.

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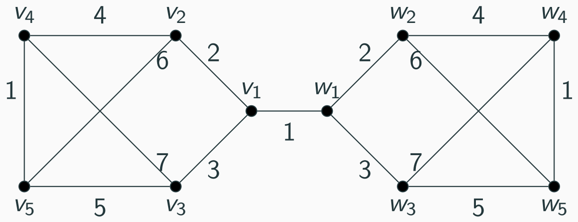

Figure 12: A cubic graph that requires 7 colors in a normal coloring.

The bridge is poor. All other edges are rich. It can be shown that $\chi_N(G) = 7$.

An example of a cubic graph without a normal k-edge-coloring

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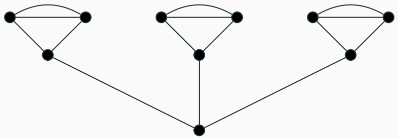

Figure 13: An example of a cubic graph that does not admit a normal coloring.

Why normal colorings are important

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Proposition (Jaeger, 1988)

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A cubic graph G admits a P_{10}-coloring (i.e. $P_{10} \prec G$), if and only if $\chi'_N(G) \leq 5$.

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Our result

A question

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Theorem (Our main result)

For any simple cubic graph G, we have $\chi'_N(G) \leq 7$.

Future work

Improving our result

Improving our result

Bridgeless cubic graphs

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- Proving $\chi'_N(G) \leq 7$ in the class of bridgeless cubic graphs is relatively easy (8-flow theorem).

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Conjecture (Intermediate conjecture)

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Conjecture (Intermediate conjecture)

For any bridgeless cubic graph G, we have $\chi'_N(G) \leq 6$.

THANK YOU!