(1) Show that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

(2) Show that if a, b, c and d are integers, where $a \neq 0$, such that $a \mid c$ and $b \mid d$, then $ab \mid cd$.

(3) Show that if a, b and c are integers, where $a \neq 0$ and $c \neq 0$, such that $ac \mid bc$, then $a \mid b$.

(4) Prove or disprove that if $a \mid bc$, where a, b and c are positive integers and $a \neq 0$, then $a \mid b$ or $a \mid c$.

(5) What are the quotient and remainder when 1001 is divided by 13?

(6) What are the quotient and remainder when -123 is divided by 19?

(7) What are the quotient and remainder when 0 is divided by 17?

(8) What time does a 24-hour clock read 100 hours after it reads 2:00?

(9) Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \leq c \leq 12$ such that

(a) $c \equiv 9a \pmod{13}$
(b) $c \equiv 11b \pmod{13}$
(c) $c \equiv a + b \pmod{13}$
(d) $c \equiv 2a + 3b \pmod{13}$
(e) $c \equiv a^2 + b^2 \pmod{13}$
(f) $c \equiv a^3 - b^3 \pmod{13}$

(10) Show that if n is an integer, then $n^2 \equiv 0$ or $1 \pmod{4}$.

(11) Use the previous exercise to show that if m is a positive integer of the form $4k + 3$ for some nonnegative integer k, then m is not the sum of the squares of two integers.

(12) Prove that if n is an odd positive integer, then $n^2 \equiv 1 \pmod{8}$.

(13) Write out the addition and multiplication tables for $\mathbb{Z}_6$ (where by addition and multiplication we mean $+_6$ and $\cdot_6$). Is $\mathbb{Z}_6$ a group under addition modulo 6? If it is, is it abelian (commutative)? Is $\mathbb{Z}_6$ a group under multiplication modulo 6?

(14) Show that $\mathbb{Z}_m$ with addition modulo m, where $m \geq 2$ is an integer, satisfies the closure, associative, and commutative properties, and 0 is an additive identity, and for every nonzero $a \in \mathbb{Z}_m$, $m - a$ is an inverse of a modulo m.

(15) Is $\mathbb{Z}_m$ a group under multiplication modulo m, where $m \geq 2$ is an integer?

(16) Let $\mathbb{Z}_7^{\times} = \mathbb{Z}_7 \setminus \{0\}$. Is $\mathbb{Z}_7^{\times}$ a group under multiplication modulo 7?

(17) Let $\mathbb{Z}_8^{\times} = \{x \in \mathbb{Z}_8 \mid \gcd(x, 8) = 1\}$. Is $\mathbb{Z}_8^{\times}$ a group under multiplication modulo 8?

(18) Determine whether each of these integers is prime.

(a) 21
(b) 29
(c) 71
(d) 97
(e) 111
(f) 143

(19) Find the prime factorization of each of these integers.

(a) 88
(b) 126
(c) 729
(d) 1001
(e) 1111
(f) 909,090

(20) Find the prime factorization of 10!

(21) Prove or disprove that there are three consecutive odd positive integers that are primes, that is, odd primes of the form p, $p+2$, and $p+4$.

(22) Determine whether the integers in each of these sets are pairwise relatively prime.

(a) 21, 34, 55
(b) 14, 17, 85
(c) 25, 41, 49, 64
(d) 17, 18, 19, 23

(23) Which positive integers less than 30 are relatively prime to 30?

(24) The value of the **Euler ϕ-function** at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. [Note: ϕ is the Greek letter phi.]
(a) Find $\phi(4)$, $\phi(10)$ and $\phi(13)$.
(b) Show that n is prime if and only if $\phi(n) = n - 1$.
(c) What is the value of $\phi(p^k)$ when p is prime and k is a positive integer.

(25) Find the greatest common divisor of the pair of integers $3^7 \cdot 5^3 \cdot 7^3$, $2^{11} \cdot 3^5 \cdot 5^9$.

(26) Find the least common multiple of the pair of integers $3^7 \cdot 5^3 \cdot 7^3$, $2^{11} \cdot 3^5 \cdot 5^9$.

(27) (a) Find $\gcd(1000, 625)$ and $\text{lcm}(1000, 625)$ and verify that $\gcd(1000, 625) \cdot \text{lcm}(1000, 625) = 1000 \cdot 625$.
(b) Show that if a and b are positive integers, then $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$. [Hint: Use the prime factorizations of a and b and the formulae for $\gcd(a, b)$ and $\text{lcm}(a, b)$ in terms of these factorizations.]

(28) Use the Euclidean algorithm to find

(a) $\gcd(1, 5)$
(b) $\gcd(100, 101)$
(c) $\gcd(123, 277)$
(d) $\gcd(1529, 14039)$
(e) $\gcd(1529, 14038)$
(f) $\gcd(11111, 111111)$

(29) Express the greatest common divisor of each of the pairs of integers in the previous problem as a linear combination of those integers.

(30) Prove or disprove that $n^2 - 79n + 1601$ is prime whenever n is a positive integer.