(1) Prove that
$$1 + 2 + 2^2 + \cdots + 2^n \; = \; 2^{n+1} - 1$$
for all nonnegative integers n.

(2) Prove that the inequality $n < 2^n$ holds for all positive integers n.

(3) Prove that $2^n < n!$ for every integer n with $n \geq 4$.

(4) Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:
$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots ar^n \; = \; \frac{ar^{n+1} - a}{r - 1} \quad \text{when } r \neq 1,$$
where n is a nonnegative integer.

(5) The **harmonic numbers** $H_j, j = 1, 2, 3, \ldots,$ are defined by
$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$
Use mathematical induction to show that
$$H_{2^n} \geq 1 + \frac{n}{2},$$
whenever n is a nonnegative integer.

(6) Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

(7) Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

(8) Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

(9) Use mathematical induction to prove the following generalization of one of De Morgan's laws:
$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

(10) (a) Find a formula for
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$
by examining the values of this expression for small values of n.

(b) Prove the formula you conjectured in part (a).