The Pigeonhole Principle

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof

Note The pigeonhole principle appeared in 1624 in a book attributed to *Jean Leurechon*, who was a French Jesuit priest, astronomer, and mathematician. The pigeonhole principle is also called the **Dirichlet drawer principle**, after the nineteenth-century German mathematician *G. Lejeune Dirichlet*, who often used this principle in his work.

Example 1 Among any group of 367 people, there must be at least two with the same birthday because there are only 366 possible birthdays.

Example 2 In any group of 27 English words, there must be at least two that begin with the same letter because there are only 26 letters in the English alphabet.

Example 3 How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/K \rceil$ objects.

Proof

Example 4 Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example 5 What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Example 6

(a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

(b) How many must be selected to guarantee that at least three hearts are selected?

Example 7 What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form $NXX-NXX-XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Example 8 During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Example 9 Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Example 10 In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

Example 11

(a) Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

(b) Let d be a positive integer. Show that among any group of $d+1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

(c) Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.