Due by 8:05am on Monday, May 1

(1) (5 points each)

(a) Prove that if a, b, and c are positive integers such that $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

(b) Let m be a positive integer and let a, b, and c be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

(2) (10 points) **Euclid's Lemma** Let p be a prime, and let a, b be integers. if $p \mid ab$, then $p \mid a$ or $p \mid b$.

Prove Euclid's Lemma.

(3) (5 points each) We call a positive integer **perfect** if it equal the sum of its positive divisors other than itself.

(a) Show that 6 and 28 are perfect.

(b) Show that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is prime.

(4) (5 points each)

(a) What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

(b) Give a formula for the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$, where k is an integer.

(5) (5 points each)

(a) Find the expansion of $(x + y)^6$ using the binomial theorem.

(b) Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

(6) (5 points each)

(a) Prove that for all real numbers x and y,

$$(-x) \cdot y \;=\; x \cdot (-y) \;=\; -(x \cdot y)$$

(b) Prove that for all real numbers x and y,

$$-(x + y) \;=\; (-x) + (-y)$$

(7) (5 points each)

(a) Prove that for all real numbers x and y,

$$(-x) \cdot (-y) \;=\; x \cdot y$$

(b) Prove that for all real numbers x, y, and z, if $x + z = y + z$, then $x = y$.

(8) (5 points each)

(a) Prove that for all real numbers x and y, if $x > 0$ and $y < 0$, then $x \cdot y < 0$.

(b) Prove that for all positive real numbers x and y, if $x < y$, then $1/x > 1/y$.

(9) (10 points) Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

(10) (10 points) Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.