Due by 8:05am on Friday, March 31

(1) Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) $\{(f,g) \mid f(1) = g(1)\}$.

(b) $\{(f,g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$.

(c) $\{(f,g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$.

(d) $\{(f,g) \mid \text{ for some } C \in \mathbb{Z}, \text{ for all } x \in \mathbb{Z},\ f(x) - g(x) = C\}$.

(e) $\{(f,g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$.

(2) Determine whether the relation R on the set A is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive.

(a) $A = \mathbb{Z}$; aRb if and only if $a \leq b + 1$.
(b) $A = \mathbb{Z}^+$; aRb if and only if $|a - b| \leq 2$.
(c) $A = \mathbb{Z}^+$; aRb if and only if $a = b^k$ for some $k \in \mathbb{Z}^+$.
(d) $A = \mathbb{Z}$; aRb if and only if $a + b$ is even.
(e) $A = \mathbb{Z}$; aRb if and only if $|a - b| = 2$.
(f) $A = \mathbb{R}$; aRb if and only if $a^2 + b^2 = 4$.
(g) $A = \mathbb{Z}^+$; aRb if and only if $\gcd(a, b) = 1$. In this case, we say that a and b are **relatively prime**.
(h) $A =$ the set of all ordered pairs of real numbers; $(a, b)R(c, d)$ if and only if $a = c$.
(i) $S = \{1, 2, 3, 4\}$; $A = S \times S$; $(a, b)R(c, d)$ if and only if $ad = bc$.

(3) (a) Show that if a relation R on a set A is transitive and irreflexive, then it is asymmetric.
(b) A relation R on a set A is called **circular** if aRb and bRc imply cRa. Show that R is reflexive and circular if and only if it is an equivalence relation.
(c) Show that if R_1 and R_2 are equivalence relations on A, then $R_1 \cap R_2$ is an equivalence relation on A.

(4) Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x - y$ is even. Determine whether $\sim$ is an equivalence relation on $\mathbb{Z}$. If it is, find the equivalence classes.

(5) Define a relation $\sim$ on $\mathbb{R}$ as follows: For $x, y \in \mathbb{R}$, we say $x \sim y$ if and only if $x^2 - y^2 \in \mathbb{Z}$.

(a) Prove that $\sim$ defined above is an equivalence relation on $\mathbb{R}$.

(b) Give five different real numbers that are in the equivalence class $[\sqrt{2}]$.

(6) Define a relation $\sim$ on $\mathbb{R}^2$ as follows: For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we say that $(x_1, x_2) \sim (y_1, y_2)$ if and only if both $x_1 - y_1$ and $x_2 - y_2$ are even integers. Is this relation an equivalence relation? Why or why not?