Please do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

MATH 243 Spring 2023 Final Exam

May 11, 2023

Your name____

The exam has 12 different printed sides of exam problems and 1 side workspace.

Duration of the Final Exam is two and a half hours. There are 12 problems, 10 points each. Only 10 problems will be graded. If you solve more than 10 problems, you must cross out the problem(s) in the box above that must not be graded. If you solve more than 10 problems and do not cross out problems, only the first ten problems will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1) Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences **and** using a truth table.

2) Let $Q(x, y)$ be the statement "$x + y = x - y$". If the domain for both variables consists of all integers, what are these truth values?

(i) $Q(1, 1)$

(ii) $\forall y\, Q(1, y)$

(iii) $\exists x\, Q(x, 2)$

(iv) $\forall x\, \exists y\, Q(x, y)$

(v) $\forall x\, \forall y\, Q(x, y)$

3) Let A and B be subsets of the set X.

(a) Prove that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

(b) Prove that $A \cap B = \emptyset$ if and only if $B \subseteq (X \setminus A)$.

4)

(a) Use a proof by contradiction to prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

(b) Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

5)

(a) Determine whether the relation R on $\mathbb{Z}^+$ is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive.

$$aRb \text{ if and only if } \gcd(a, b) = 1.$$

(b) Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x - y$ is even. Determine whether $\sim$ is an equivalence relation on $\mathbb{Z}$. If it is, find the equivalence classes.

6)

(a) Use the Euclidean algorithm to find $\gcd(1529, 14038)$.

(b) Express the greatest common divisor of the pair of integers in part (a) as a linear combination of those integers.

7)

(a) State and prove Euclid's Lemma.

(b) Prove that there are infinitely many primes.

8)

(a) What is the coefficient of $x^{18}y^{16}$ in the expansion of $(3x - \frac{1}{3}y)^{34}$?

(b) Compute $(x+y)^3 \pmod 3$, $(x+y)^5 \pmod 5$, and $(x+y)^7 \pmod 7$.

(c) Looking at the expansions in part (b), conjecture a formula for $(x+y)^p \pmod p$, where p is prime. Use the Binomial Theorem to justify your answer.

9)

(a) Let $\mathbb{Z}_{14}^{\times} = \{x \in \mathbb{Z}_{14} \,|\, \gcd(x, 14) = 1\}$. Write out the multiplication table and prove that $\mathbb{Z}_{14}^{\times}$ is an abelian group under multiplication modulo 14.

(b) Let $\mathbb{Z}_{72}^{\times} = \{x \in \mathbb{Z}_{72} \,|\, \gcd(x, 72) = 1\}$. Find the number of elements of the set $\mathbb{Z}_{72}^{\times}$.

10)

(a) Prove that for all real numbers x and y, $-(x+y) = (-x) + (-y)$.

(b) Prove the **Archimedean Property**: For every real number x, there exists an integer n such that $n > x$.

11) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x + 5$.

(a) Is f one-to-one?

(b) Is f onto?

(c) Find the range of f.

12)

(a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9. Is the conclusion true if four integers are selected rather than five?

(b) In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?

Workspace