(1) Let (X, Y) denote the coordinates of a point chosen at random inside a unit circle whose center is at the origin. That is, X and Y have a joint density function given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

$P(X \le Y)$.

(2) Show that

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

is a valid joint pdf. What can you say about $\text{Cov}(X, Y)$? Are X and Y independent? Find $E(X-Y)$ and $\text{Var}(X-Y)$.

(3) Show that

$$f_{X,Y}(x,y) = \begin{cases} 6(1-y) & 0 \le x \le y \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

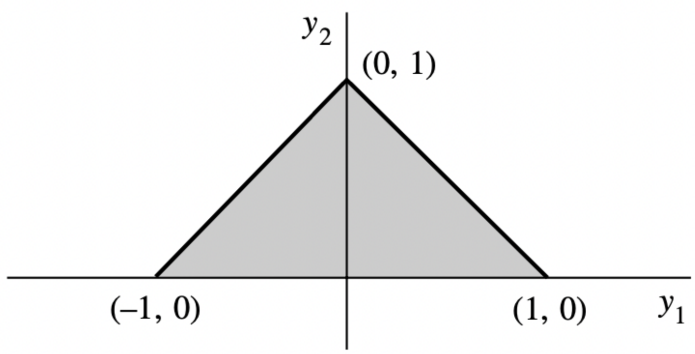
is a valid joint pdf. Find $\text{Cov}(X, Y)$, $E(X - 3Y)$ and $\text{Var}(X - 3Y)$.

(4) Show that

$$p(x,y) = \frac{\binom{4}{x}\binom{3}{y}\binom{2}{3-x-y}}{\binom{9}{3}},$$

where $0 \le x \le 3$, $0 \le y \le 3$, and $1 \le x + y \le 3$, is a joint pdf.
(a) Find $\text{Cov}(X, Y)$.
(b) Find $E(X + Y)$ and $\text{Var}(X + Y)$.

(5) Random variables X and Y are uniformly distributed over the triangle shaded in the accompanying diagram.



(a) Find $P(X \le 3/4, Y \le 3/4)$
(b) Find $P(X - Y \ge 0)$
(c) Find $\text{Cov}(X, Y)$.
(d) Are X and Y independent?
(e) Find the correlation coefficient for X and Y.
(f) Does your answer to part (b) lead you to doubt your answer to part (a)? Why or why not?

(6) Let the discrete random variables X and Y have the joint probability function

$$p_{X,Y}(x,y) = 1/3, \quad \text{for } (x,y) = (-1,0), (0,1), (1,0).$$

Find $\text{Cov}(X, Y)$. Notice that X and Y are dependent. (Why?) This is an example of uncorrelated random variables that are not independent.

(7) Let X and Y be uncorrelated random variables and consider $U_1 = X + Y$ and $U_2 = X - Y$.

(a) Find the $\text{Cov}(U_1, U_2)$ in terms of the variances of X and Y.
(b) Find an expression for the coefficient of correlation between U_1 and U_2.
(c) Is it possible that $\text{Cov}(U_1, U_2) = 0$? When does this occur?

(8) Suppose that

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)},\ 0 \le x,\ 0 \le y.$$

Find $\text{Var}(X+Y)$, $E(X+Y)$, and $\text{Cov}(X,Y)$.

(9) Let X and Y denote the proportions of time (out of one workday) during which employees I and II, respectively, perform their assigned tasks. The joint relative frequency behavior of X and Y is modeled by the density function

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $P(X < 1/2,\, Y > 1/4)$.
(b) Find $P(X + Y \le 1)$.
(c) Find the marginal density functions for X and Y.
(d) Find $(X \ge 1/2 \,|\, Y \ge 1/2)$
(e) If employee II spends exactly 50% of the day working on assigned duties, find the probability that employee I spends more than 75% of the day working on similar duties.
(f) Are X and Y independent?
(g) Employee I has a higher productivity rating than employee II and a measure of the total productivity of the pair of employees is $30X + 25Y$. Find the expected value of this measure of productivity.
(h) Find the variance of this measure of productivity $30X + 25Y$. Give an interval in which you think the total productivity measures of the two employees should lie for at least 75% of the days in question.

(10) The random variables X and Y are such that $E(X) = 4$, $E(Y) = -1$, $\text{Var}(X) = 2$ and $\text{Var}(Y) = 8$.
(a) What is $\text{Cov}(X,X)$?
(b) Assuming that the means and variances are correct, as given, is it possible that $\text{Cov}(X,Y) = 7$? [Hint: If $\text{Cov}(X,Y) = 7$, what is the value of ρ, the coefficient of correlation?]
(c) Assuming that the means and variances are correct, what is the largest possible value for $\text{Cov}(X,Y)$? If $\text{Cov}(X,Y)$ achieves this largest value, what does that imply about the relationship between X and Y?
(d) Assuming that the means and variances are correct, what is the smallest possible value for $\text{Cov}(X,Y)$? If $\text{Cov}(X,Y)$ achieves this smallest value, what does that imply about the relationship between X and Y?

(11) Let Z be a standard normal random variable and let $Y = Z$ and $Y = Z^2$.
(a) What are $E(X)$ and $E(Y)$?
(b) What is $E(XY)$?
(c) What is $\text{Cov}(X,Y)$?
(d) Are X and Y independent?

(12) Suppose that X and Y have correlation coefficient $\rho = 0.2$. What is the value of the correlation coefficient between $1 + 2X$ and $3 + 4Y$?