Definition Suppose S is a discrete sample space on which two random variables, X and Y, are defined. The *joint probability density function* of X and Y (or joint pdf) is denoted $p_{X,Y}(x, y)$, where

$$p_{X,Y}(x, y) = P\Big(\{s \,|\, X(s) = x \text{ and } Y(s) = y\}\Big).$$

We simply write $p_{X,Y}(x, y) = P(X = x, Y = y)$.

Theorem If X and Y are discrete random variables with joint pdf $p_{X,Y}(x, y)$, then

(1) $p_{X,Y}(x, y) \geq 0$ for all x and y

(2) $\displaystyle\sum_{x,y} p_{X,Y}(x, y) = 1$, where the sum is over all values (x, y) that are assigned nonzero probabilities.

Theorem Suppose that $p_{X,Y}(x, y)$ is the joint pdf of the discrete random variables X and Y. Then

$$p_X(x) = \sum_{\text{all } y} p_{X,Y}(x, y) \quad \text{and} \quad p_Y(y) = \sum_{\text{all } x} p_{X,Y}(x, y).$$

Definition An individual pdf obtained by summing a joint pdf over all values of the other random variable is called a *marginal pdf.*

Note: For examples on discrete joint pdfs, please refer to Worksheet 9.

Definition Two random variables defined on the same set of real numbers are *jointly continuous* if there exists a function $f_{X,Y}(x, y)$ such that for any region R in the xy-plane, $P[(X, Y) \in R] = \displaystyle\int\int_R f_{X,Y}(x, y)\, dx\, dy$. The function , $f_{X,Y}(x, y)$ is the *joint pdf* of X and Y.

Comment Any function $f_{X,Y}(x, y)$ for which

(1) $f_{X,Y}(x, y) \geq 0$ for all x and y

(2) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$

qualifies as a joint pdf.

Exercise Suppose that the variation in two continuous random variables, X and Y, can be modeled by the joint pdf $f_{X,Y}(x, y) = cxy$, for $0 < y < x < 1$. Find c.

Exercise A study claims that the daily number of hours X, a teenager watches television and the daily number of hours, Y, he works on his homework are approximated by the joint pdf

$$f_{X,Y}(x,y) = xye^{-(x+y)}, \quad x > 0, \quad y > 0$$

What is the probability that a teenager chosen at random spends at least twice as much time watching television as he does working on his homework?

The Joint Uniform Pdf

$$f_{X,Y}(x,y) = \frac{1}{(b-a)(d-c)}, \quad a \le x \le b, \; c \le y \le d$$

Geometrically, the pdf defines a surface having a constant height everywhere above a specified rectangle in the xy-plane. If R is some region in the rectangle where X and Y are defined, $P((X,Y) \in R)$ reduces to a simple ratio of areas:

$$P((X,Y) \in R) = \frac{\text{area of } R}{(b-a)(d-c)}$$

Calculations based on this equation are referred to as *geometric probabilities.*

Example Two friends agree to meet on the University Commons "sometime around 12:30." But neither of them is particularly punctual—or patient. What will actually happen is that each will arrive at random sometime in the interval from 12:00 to 1:00. If one arrives and the other is not there, the first person will wait fifteen minutes or until 1:00, whichever comes first, and then leave. What is the probability that the two will get together?

Marginal Pdfs for Continuous Random Variables

Theorem Suppose X and Y are jointly continuous with joint pdf $f_{X,Y}(x, y)$. Then the marginal pdfs, $f_X(x)$ and $f_Y(y)$, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

Exercise Suppose that two continuous random variables, X and Y, have the joint uniform pdf

$$f_{X,Y}(x, y) = \frac{1}{6}, \quad 0 \le x \le 3, \quad 0 \le y \le 2.$$

Find $f_X(x)$.

Exercise Consider the case where X and Y are two continuous random variables, jointly distributed over the first quadrant of the xy-plane according to the joint pdf,

$$f_{X,Y}(x, y) = \begin{cases} y^2 e^{-y(x+1)} & \text{if } x \ge 0, \quad y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the two marginal pdfs.

Joint Cdfs

Definition Let X and Y be any two random variables. The *joint cumulative distribution function of X and Y* (or joint cdf) is denoted $F_{X,Y}(u,v)$, where

$$F_{X,Y}(u,v) = P(X \le u \text{ and } Y \le v)$$

Discrete Case

$$F_{X,Y}(u,v) = \sum_{x \le u} \sum_{y \le v} p(x,y)$$

Continuous Case

$$F_{X,Y}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(x,y)\; dy\, dx, \text{ for all } -\infty < u < \infty,\ -\infty < v < \infty$$

Exercise Find the joint cdf, $F_{X,Y}(u,v)$, for the two random variables X and Y whose joint pdf is given by

$$f_{X,Y}(x,y) = \frac{4}{3}(x + xy),\ 0 \le x \le 1,\ 0 \le y \le 1.$$

Theorem Let $F_{X,Y}(u,v)$ be the joint cdf associated with the continuous random variable X and Y. then the joint pdf of X and Y, $f_{X,Y}(x,y)$, is a second partial derivative of the joint cdf - that is, $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$, provided $F_{X,Y}(x,y)$ has continuous second partial derivatives.

Exercise What is the joint pdf of the random variables X and Y whose joint cdf is $F_{X,Y}(x,y) = \frac{1}{3}x^2(2y + y^2)$?

Independence of Two Random Variables

Definition Two random variables X and Y are said to be *independent* if for every interval A and every interval B, we have

$$P(X \in A \text{ and } Y \in B) = P(X \in A) \cdot P(Y \in B).$$

If x and Y are not independent, they are said to be *dependent*.

Theorem Let X and Y be discrete random variables with joint pdf $p_{X,Y}(x, y)$ and marginal pdfs $p_X(x)$ and $p_Y(y)$, respectively. Then X and Y are independent if and only if

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y).$$

Theorem Let X and Y be continuous random variables with joint pdf $f_{X,Y}(x, y)$ and marginal pdfs $f_X(x)$ and $f_Y(y)$, respectively. Then X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y).$$

Exercise Suppose that the probabilistic behavior of two random variables X and Y is described by the joint pdf

$$f_{X,Y}(x, y) = 12xy(1 - y),\ 0 \leq x \leq 1,\ 0 \leq y \leq 1.$$

Are X and Y independent? If they are, find $f_X(x)$ and $f_Y(y)$.

Chebyshev's Inequality (Chebyshev's Theorem)

Let X be a random variable with finite mean μ and variance σ^2. Then, for any $k > 0$,

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \text{ or } P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Note: The first inequality gives you a lower bound for the probability that any random variable lies inside an interval of length $2k\sigma$ centered around the mean. That is, the probability that any random variable lies in the interval $(\mu - k\sigma, \mu + k\sigma)$. The second inequality gives you an upper bound for the probability that any random variable lies outside that interval.

Exercise Suppose that experience has shown that the length of time Y (in minutes) required to conduct a periodic maintenance check on a dictating machine follows a gamma distribution with $\alpha = 3.1$ and $\beta = 2$. A new maintenance worker takes 22.5 minutes to check the machine. Does this length of time to perform a maintenance check disagree with prior experience?

Further Properties of the mean and variance

Theorem Suppose X and Y are discrete random variables with joint pdf $p_{X,Y}(x,y)$, and let $g(X,Y)$ be a function of X and Y. Then the expected value of the random variable $g(X,Y)$ is given by

$$E[g(X,Y)] = \sum_{\text{all } x} \sum_{\text{all } y} g(x,y) \cdot p_{X,Y}(x,y)$$

provided $\sum_{\text{all } x} \sum_{\text{all } y} |g(x,y)| \cdot p_{X,Y}(x,y) < \infty$.

Theorem Suppose X and Y are continuous random variables with joint pdf $f_{X,Y}(x,y)$, and let $g(X,Y)$ be a function of X and Y. Then the expected value of the random variable $g(X,Y)$ is given by

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f_{X,Y}(x,y)$$

provided $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y)| \cdot f_{X,Y}(x,y) < \infty$.

Exercise Consider two random variables X and Y whose joint pdf is detailed in the following table. Let

$$g(X,Y) = 3X - 2XY + Y.$$

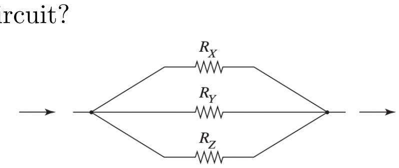
Find $E[g(X,Y)]$.

X \ Y	0	1	2	3
0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

Example A electrical circuit has three resistors, R_X, R_Y, and R_Z, wired in parallel (see below). The nominal resistance of each is fifteen ohms, but their *actual* resistances, X, Y, and Z, vary between ten and twenty according to the joint pdf

$$f_{X,Y,Z}(x,y,z) = \frac{1}{675,000}(xy + xz + yz), \quad 10 \le x, y, z \le 20.$$

What is the expected resistance for the circuit?



Let R denote the circuit's resistance. A well-known result in physics holds that

$$\frac{1}{R} = \frac{1}{X} + \frac{1}{Y} + \frac{1}{Z}.$$

Theorem Let X and Y be any two random variables (discrete or continuous, dependent or independent), and let a and b be any two constants. Then

$$E(aX + bY) = aE(X) + bE(Y)$$

provided $E(X)$ and $E(Y)$ are both finite.

Note: This result can be generalized to n random variables provided that the expected value of each random variable is finite.

Exercise: L:et X be a binomial random variable defined on n independent trials, each trial resulting in success with probability p. Then X can be thought of as a sum of n *Bernoulli* random variables:

$$X = X_1 + X_2 + \cdots + X_n,$$

where $X_i \sim \text{Bernouli}(p)$. Then $E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot p = np$. This is a result we proved in class using the definition of the expected value of a random variable.

Exercise: A disgruntled secretary is upset about having to stuff envelopes. Handed a box of n letters and n envelopes, she vents her frustration by putting the letters into the envelopes at random. How many people, on the average, will receive their correct mail?

Exercise: Ten fair dice are rolled. Calculate the expected value of the sum of the faces showing.

Expected Values of Products: A Special Case

Question: How do we compute the expected value of the product XY? In the definition of $E[g(X, Y)]$, we can take $g(X, Y)$ to be XY. When X and Y are independent, there is an easier way to compute $E(XY)$.

Theorem If X and Y are independent random variables,

$$E(XY) = E(X) \cdot E(Y)$$

provided $E(X)$ and $E(Y)$ both exist.

When random variables are not independent, a measure of the relationship between them, their *covariance*, enters into the picture.

Definition Given random variables X and Y with means μ_X and μ_Y, define the *covariance* of X and Y, written $\text{Cov}(XY)$, as

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - E(X)E(Y).$$

Theorem If X and Y are independent, then $\text{Cov}(X, Y) = 0$.

Proof The proof follows from the fact that $E(XY) = E(X) \cdot E(Y)$ since X and Y are independent.

Note: The converse of the previous theorem is *not* true. Just because $\text{Cov}(X, Y) = 0$, we cannot conclude that X and Y are independent. Here is an example.

Exercise Consider the sample space $S = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$, where each point is assumed to be equally likely. Define the random variable X to be the first component of a sample point and Y, the second. Then $X(-2, 4) = -4$, $Y(-2, 4) = 4$, and so on. Show that $\text{Cov}(X, Y) = 0$, but X and Y are dependent.

Exercise Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let X denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, X varies from week to week. Let Y denote the proportion of the capacity of the bulk tank that is sold during the week. Because X and Y are both proportions, both variables take on values between 0 and 1. Further, the amount sold, y, cannot exceed the amount available, x. Suppose that the joint density function for X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} 3x & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the covariance between the amount in stock X and amount of sales Y.

The following theorem demonstrates the role of the covariance in finding the variance of a sum of random variables that are not necessarily independent.

Theorem Suppose X and Y are random variables with finite variances, and a and b are constants. Then

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab\, \text{Cov}(X, Y).$$

Proof Exercise

Exercise For the joint pdf $f_{X,Y}(x, y) = x + y$, $0 \le x \le 1$, $0 \le y \le 1$, find the variance of $X + Y$.

Hint: Note that X and Y are not independent as the joint pdf cannot be written as a product of function in x alone and a function in y alone. Also, note that the joint pdf is symmetric in x and Y, so $\text{Var}(X) = \text{Var}(Y)$. Find the marginal pdf of X, and then find $E(X)$, $E(X^2)$.

The previous result can be extended to n random variables with finite variances.

Corollary Suppose that $X_1, X_2, \ldots, X_n$ are random variables with finite variances. Then

$$\text{Var}\Big(\sum_{i=1}^{n} a_i X_i\Big) = \sum_{i=1}^{n} a_i^2\, \text{Var}(X_i) + 2 \sum_{i<j} a_i a_j\, \text{Cov}(X_i, X_j).$$

Corollary Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with finite variances. Then

$$\text{Var}(X_1 + X_2 + \cdots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n).$$

Exercise Can you use the result in the second corollary to derive the formula for the variance of a Binomial random variable?

Let $X_1, X_2, \ldots, X_n$ be independent random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. (These variables may denote the outcomes of n independent trials of an experiment.) Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Proof The proof follows from a previous theorem and a corollary.

Exercise The number of defectives Y in a sample of $n = 10$ items selected from a manufacturing process follows a binomial probability distribution. An estimator of the fraction defective in the lot is the random variable $\hat{p}$. Find the expected value and variance of $\hat{p}$.

Central Limit Theorem Let $X_1, X_2, \ldots, X_n$ be independent random variables, each with the same distribution. Suppose that $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$. Then the distribution of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ converges to the standard normal distribution as $n \to \infty$. That is, For any numbers a and b,

$$\lim_{n\to\infty} P\Big(a \leq \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq b\Big) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-z^2/2}\,dz,$$

which is equivalent to saying

$$\lim_{n\to\infty} P\Big(a \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq b\Big) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-z^2/2}\,dz,$$

which is equivalent to saying

$$\lim_{n\to\infty} P\Big(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq u\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^u e^{-z^2/2}\,dz \quad \text{for all } u.$$

Exercise Achievement test scores of all high school seniors in a state have mean 60 and variance 64. A random sample of $n = 100$ students from one large high school had a mean score of 58. Is there evidence to suggest that this high school is inferior? (Calculate the probability that the sample mean is at most 58 when $n = 100$.)

Exercise The service times for customers coming through a checkout counter in a retail store are independent random variables with mean 1.5 minutes and variance 1.0. Approximate the probability that 100 customers can be served in less than 2 hours of total service time.

Exercise A random sample of size $n = 15$ is drawn from the pdf $f_Y(y) = 3(1-y)^2$, $0 \leq y \leq 1$. Let $\overline{Y} = \frac{1}{15}\sum_{i=1}^{15} Y_i$. Use the central limit theorem to approximate $P(\frac{1}{8} \leq \overline{Y} \leq \frac{3}{8})$.

Exercise In preparing next quarter's budget, the accountant for a small business has one hundred different expenditures to account for. Her predecessor listed each entry to the penny, but doing so grossly overstates the precision of the process. As a more truthful alternative, she intends to record each budget allocation to the nearest \$100. What is the probability that her total estimated budget will end up differing from the actual cost by more than \$500? Assume that $Y_1, Y_2, \ldots, Y_100$, the rounding errors she makes on the one hundred items, are independent and uniformly distributed over the interval $[-\$50, +\$50]$.