Due by 4pm on Friday, October 4. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code. Each problem is worth 10 points.

(1) (5 points each)

(a) How many probability equations need to be verified to establish the mutual independence of *five* events?

(b) Suppose that A and B are independent events such that the probability that neither occurs is a and the probability of B is b. Show that $P(A) = \dfrac{1-b-a}{1-b}$.

(2) (5 points each)

(a) Show that the additive law of probability holds for conditional probabilities. That is, if A, B, and C are events such that $P(C) > 0$, prove that

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C).$$

(b) Let A, B, and C be events such that $P(A) > P(B)$ and $P(C) > 0$. Construct an example to demonstrate that it is possible that $P(A|C) < P(B|C)$.

(3) (5 points each)

(a) An urn contains w white chips, b black chips, and r red chips. The chips are drawn out at random, one at a time, with replacement. What is the probability that a white appears before a red?

(b) Players A, B, and C toss a fair coin in order. The first to throw a head wins. What are their respective chances of winning?

(4) (5 points each)

(a) The crew of Apollo 17 consisted of a pilot, a copilot, and a geologist. Suppose that NASA had actually trained nine aviators and four geologists as candidates for the flight. How many different crews could they have assembled?

(b) A three-digit number is to be formed from the digits 1 through 7, with no digit being used more than once. How many such numbers would be less than 379?

(c) A deck of fifty-two cards is shuffled and dealt face up in a row. For how many arrangements will the four aces be adjacent?

(d) In her sonnet with the famous first line, “How do I love thee? Let me count the ways,” Elizabeth Barrett Browning listed eight ways. Suppose Ms. Browning had decided that writing greeting cards afforded her a better format for expressing her feelings. For how many years could she have corresponded with her favorite beau on a daily basis and never sent the same card twice? Assume that each card contains exactly four of the eight “ways” and that order matters.

(5) (5 points each)

(a) What is the coefficient of x^{23} in the expansion of $(1 + x^5 + x^9)^{100}$?

(b) What is the coefficient of $v^3w^2x^3yz^3$ in the expansion of $(v + w + x + y + z)^{11}$?

(c) Which state name can generate more permutations, TENNESSEE or FLORIDA?

(d) A tennis tournament has a field of $2n$ entrants, all of whom need to be scheduled to play in the first round. How many different pairings are possible?

(6) (5 points each)

(a) Ten basketball players meet in the school gym for a pickup game. How many ways can they form two teams of five each?

(b) In how many ways can the letters in MISSISSIPPI be arranged so that no two I's are adjacent?

(c) Prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(d) Show that

$$\binom{n}{1} + \binom{n}{3} + \cdots = \binom{n}{0} + \binom{n}{2} + \cdots$$

(7) (10 points) Let $S = \{1, 2, 3\}$, $X = I_{\{1\}}$, $Y = I_{\{2,3\}}$, and $Z = I_{\{1,2\}}$. Let $W = X - Y + Z$.

(a) Compute $W(1)$.

(b) Compute $W(2)$.

(c) Compute $W(3)$.

(d) Determine whether or not $W \geq Z$.