COLLEGE OF THE HOLY CROSS
Department of Mathematics and Computer Science

STAT 375 Probability Theory
Fall 2024 Final Exam

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Points	10	10	8	8	8	8	8	8	8	8	8	8	8	8	100

Your name____

Duration of the Final Exam is 150 minutes. There are 14 problems. ***The first problem and the second problem are mandatory****. From Problems 3 – 14, only 10 problems will be graded. If you solve all Problems 3 – 14, you must cross out the two problems in the boxes above that must not be graded. If you solve all Problems 3 – 14 but do not cross out two problems, only the first ten problems from 3 – 14 will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.*

1. A businesswoman in Philadelphia is preparing an itinerary for a visit to six major cities. The distance traveled, and hence the cost of the trip, will depend on the order in which she plans her route.

   (a) How many different itineraries (and trip costs) are possible?

   (b) If the businesswoman randomly selects one of the possible itineraries and Denver and San Francisco are two of the cities that she plans to visit, what is the probability that she will visit Denver before San Francisco?

2. (a) If X has moment-generating function $M_X(t) = [(1/5)e^t + (4/5)]^8$, what is $P(X \geq 2)$?

(b) If X has moment-generating function $M_X(t) = e^{3(e^t-1)}$, what is $P(0.1 \leq X \leq 4.1)$?

(c) If $M_X(t) = \dfrac{e^t}{7 - 6e^t}$ be the moment-generating function for X, find

i. $E(X)$

ii. $E(X^2)$

iii. $\text{Var}(X)$

3. Let S be a sample space, and A and B be events defined on S. Prove the following. You must mention any result, theorem or axiom used.

(i) $P(\emptyset) = 0$

(ii) If $A \subseteq B$, then $P(A) \leq P(B)$.

(iii) For any event A, $P(A) \leq 1$.

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4. Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those traveling on major airlines, 50% are traveling for business reasons, whereas 60% of those arriving on private planes and 90% of those arriving on other commercially owned planes are traveling for business reasons. Suppose that we randomly select one person arriving at this airport. What is the probability that the person

   (a) is traveling on business?

   (b) is traveling for business on a privately owned plane?

   (c) arrived on a privately owned plane, given that the person is traveling for business reasons?

   (d) is traveling on business, given that the person is flying on a commercially owned plane?

5. (a) Suppose that A and B are mutually exclusive events, with $P(A) > 0$ and $P(B) < 1$. Are A and B independent? Prove your answer.

(b) Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Are A and B independent? Prove your answer.

(c) Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

(d) If A and B are mutually exclusive events and $P(B) > 0$, show that

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

6. Students attending the College of the Holy Cross can select from 130 major areas of study. A student's major is identified in the registrar's records with a two-or three-letter code (for example, statistics majors are identified by STA, math majors by MS). Some students opt for a double major and complete the requirements for both of the major areas before graduation. The registrar was asked to consider assigning these double majors a distinct two- or three-letter code so that they could be identified through the student records' system.

   (a) What is the maximum number of possible double majors available to College of the Holy Cross students?

   (b) If any two- or three-letter code is available to identify majors or double majors, how many major codes are available?

   (c) How many major codes are required to identify students who have either a single major or a double major?

   (d) Are there enough major codes available to identify all single and double majors at the College of the Holy Cross?

7. (a) The gunner on a small assault boat fires six missiles at an attacking plane. Each has a 20% chance of being on-target. If two or more of the shells find their mark, the plane will crash. At the same time, the pilot of the plane fires ten air-to-surface rockets, each of which has a 0.05 chance of critically disabling the boat. Would you rather be on the plane or the boat?

   (b) A particular sale involves four items randomly selected from a large lot that is known to contain 10% defectives. Let X denote the number of defectives among the four sold. The purchaser of the items will return the defectives for repair, and the repair cost is given by $C = 3X^2 + X + 2$. Find the expected repair cost.

8. (a) Ten percent of the engines manufactured on an assembly line are defective. If engines are randomly selected one at a time and tested, what is the probability that the first non-defective engine will be found on the second trial? Find the mean and variance of the number of the trial on which the first non-defective engine is found.

(b) Refer to part (a). What is the probability that the third non-defective engine will be found on the fifth trial? Find the mean and variance of the number of the trial on which the third non-defective engine is found.

(c) In the daily production of a certain kind of rope, the number of defects per foot Y is assumed to have a Poisson distribution with mean $\lambda = 2$. The profit per foot when the rope is sold is given by X, where $X = 50 - 2Y - Y^2$. Find the expected profit per foot.

9. (a) The cycle time for trucks hauling concrete to a highway construction site is uniformly distributed over the interval 50 to 70 minutes. What is the probability that the cycle time exceeds 65 minutes if it is known that the cycle time exceeds 55 minutes? Find the mean and variance of the cycle times for the trucks.

(b) The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm. What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?

(c) The response times on an online computer terminal have approximately a gamma distribution with mean four seconds and variance eight seconds. Write the probability density function for the response times. Give an interval that contains at least 75% of the response times.

10. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density function

$$f_Y(y) = \begin{cases} cy^2(1-y)^4 & 0 \le y \le 1, \\ 0 & elsewhere. \end{cases}$$

(a) Find the value of c that makes $f_Y(y)$ a probability density function.

(b) Find $E(Y)$.

(c) Find $\mathrm{Var}(Y)$.

11. Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion. Let X denote the number of married executives and Y denote the number of never-married executives among the three selected for promotion.

   (a) Assuming that the three are randomly selected from the nine available, find the joint probability function of X and Y. Give the range of x, y, and $x + y$.

   (b) Find the marginal probability distribution of X, the number of married executives among the three selected for promotion.

   (c) Find $P(X = 1|Y = 2)$

   (d) Find $\text{Cov}(X, Y)$, $E(X + Y)$, and $\text{Var}(X + Y)$

12. Let X and Y denote the proportions of time (out of one workday) during which employees I and II, respectively, perform their assigned tasks. The joint relative frequency behavior of X and Y is modeled by the density function

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the marginal density functions for X and Y.

(b) Find the joint cdf $F_{X,Y}(u,v)$.

(c) Find $P(X + Y \leq 1)$.

(d) Find $(X \geq 1/2 \,|\, Y \geq 1/2)$.

13. An electronic system has one each of two different types of components in joint operation. Let X and Y denote the random lengths of life of the components of type I and type II, respectively. The joint density function is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}\, x\, e^{-(x+y)/2} & x > 0,\ y > 0, \\ 0 & \text{elsewhere} \end{cases}$$

(a) Are x and Y independent?

(b) Find $\mathrm{Cov}(X,Y)$.

(c) One way to measure the relative efficiency of the two components is to compute the ratio Y/X. Find $E(Y/X)$.

14. (a) Show that

$$f_{X,Y}(x,y) = \begin{cases} 6(1-y) & 0 \le x \le y \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

is a valid joint pdf.

(b) Find $\text{Cov}(X,Y)$.

(c) Find $\text{Var}(X + Y)$ and the correlation coefficient between X and Y.

WORKSHEET