The Uniform Probability Distribution

Definition Let $a < b$. A random variable Y is said to have a *uniform probability distribution over* $[a, b]$ if and only if

$$f_Y(y) = \frac{1}{b-a}, \quad \text{for } a \le y \le b.$$

$$\mu = E(Y) = \frac{a+b}{2} \text{ and } \sigma^2 = \text{Var}(Y) = \frac{(a-b)^2}{12}$$

The Normal Probability Distribution

Definition A random variable Y is said to have a *normal probability distribution* if and only if, for $\sigma > 0$ and $-\infty < \mu < \infty$. the density function of Y is

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad -\infty < y < \infty.$$

The normal density function contains two parameters μ and σ. The symbol $Y \sim N(\mu, \sigma^2)$ is sometimes used to denote the fact that Y has a normal distribution with mean μ and variance σ^2.

$$E(Y) = \mu \text{ and } \text{Var}(Y) = \sigma^2$$

Comment Areas under an "arbitrary" normal distribution, $f_Y(y)$, are calculated by finding the equivalent area under the standard normal distribution, $f_Z(z)$:

Let $Z = \frac{Y-\mu}{\sigma}$. Then $Z \sim N(0, 1)$.

$$P(a \le Y \le b) = P\Big(\frac{a-\mu}{\sigma} \le \frac{Y-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\Big) = P\Big(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\Big)$$

The ratio $\frac{Y-\mu}{\sigma}$ is often referred to as either a Z *transformation* or a Z *score*.

The Exponential Probability Distribution

When previously observing a process of Poisson type, we counted the number of changes occurring in a given interval. This number was a discrete-type random variable with a Poisson Distribution. But not only is the number of changes a random variable; the waiting times between successive changes are also random variables. However, the latter are of the continuous type since each of them can assume any positive value. In particular, let Y be the random variable that denotes the waiting time until the first change occurs during the observation of a Poisson process in which the mean number of changes in the unit interval is λ. Then Y is a continuous type random variable with the probability distribution function

$$f_Y(y) = \lambda e^{-\lambda y}.$$

We often let $\lambda = \frac{1}{\beta}$ and say that the random variable Y has an exponential distribution.

Definition A random variable Y is said to have an *exponential distribution* with parameter $\beta > 0$ if and only if the density function of Y is

$$f_Y(y) = \frac{1}{\beta} e^{-\frac{y}{\beta}}, \quad 0 \le y < \infty.$$

If Y has an exponential random variable with parameter β, then

$$\mu = E(Y) = \beta \text{ and } \sigma^2 = \text{Var}(Y) = \beta^2$$

The Gamma Probability Distribution

This distribution generalizes the Poisson/exponential relationship and focuses on the interval, or waiting time, required for the αth event to occur.

Theorem Suppose that Poisson events are occurring at the constant rate of $\frac{1}{\beta}$ per unit time. Let the random variable Y denote the waiting time for the αth event. Then Y has pdf $f_Y(y)$, where

$$f_Y(y) = \frac{1}{\beta^\alpha\,(\alpha-1)!}\,y^{\alpha-1}e^{-\frac{y}{\beta}},\ y > 0$$

Definition For any real number $\alpha > 0$, the *gamma function of* α is denoted $\Gamma(\alpha)$, where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$$

Definition A random variable Y is said to have a *gamma distribution with parameters* $\alpha > 0$ *and* $\beta > 0$ if and only if the density function of Y is

$$f_Y(y) = \frac{1}{\beta^\alpha\,\Gamma(\alpha)}\,y^{\alpha-1}\,e^{-\frac{y}{\beta}},\quad y > 0$$

If Y has a gamma distribution with parameters α and β, then

$$\mu = E(Y) = \alpha\beta \ \text{ and } \ \sigma^2 = \text{Var}(Y) = \alpha\beta^2$$

The Chi-Square Probability Distribution

Let $Z_1, Z_2, \ldots, Z_\nu$ be independent standard normal random variables. The pdf of $U = \sum_{j=1}^{\nu} Z_j^2$ is called the *chi-square distribution with* ν *degrees of freedom*, given by

$$f_U(u) = \frac{1}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)}\,u^{(\nu/2)-1}\,e^{-u/2},\quad u \geq 0$$

Note: If you take a closer look at the chi-square pdf, this is a special case of the gamma distribution with parameters $\alpha = \nu/2$ and $\beta = 2$. A random variable with a chi-square distribution is called a *chi-square* (χ^2) *random variable.*

If Y has a chi-square distribution with ν degrees of freedom, then

$$\mu = E(Y) = \nu \ \text{ and } \ \sigma^2 = \text{Var}(Y) = 2\nu$$

The Beta Probability Distribution

The beta density function is a two-parameter density function defined over the closed interval $0 \leq y \leq 1$. It is often used as a model for proportions, such as the proportion of impurities in a chemical product or the proportion of time that a machine is under repair.

Definition A random variable Y is said to have a *beta probability distribution with parameters* $\alpha > 0$ *and* $\beta > 0$ if and only if the density function of Y is

$$f_Y(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)},\quad 0 \leq y \leq 1,$$

where

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

If Y is a beta-distributed random variable with parameters $\alpha > 0$ and $\beta > 0$, then

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta} \ \text{ and } \ \sigma^2 = \text{Var}(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$