### Counting Combinations

**Theorem** The number of ways to form combinations of size $k$ from a set of $n$ distinct objects, repetitions not allowed, is denoted by the symbols $\binom{n}{k}$ or ${}_nC_k$, where

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}.$$

**Example 1** Eight politicians meet at a fund-raising dinner. How many greetings can be exchanged if each politician shakes hands with every other politician exactly once?

**Example 2** A chemist is trying to synthesize a part of a straight-chain aliphatic hydrocarbon polymer that consists of twenty-one-radicals – ten ethyls (E), six methyls (M), and five propyls (P). Assuming all arrangements of radicals are physically possible, how many different polymers can be formed if no two of the methyl radicals are to be adjacent?

**Example 3** Suppose you have just ordered a roast beef sub at a sandwich shop, and now you need to decide which, if any, of the available toppings (lettuce, tomato, onions, etc.) to add. If the shop has eight "extras" to choose from, how many different subs can you order?

**Example 4** Nine students, five men and four women, interview for four summer internships sponsored by a city newspaper.

(1) In how many ways can the newspaper choose a set of four interns?
(2) In how many ways can the newspaper choose a set of four interns if it must include two men and two women in each set?
(3) How many sets of four can be picked such that not everyone in a set is of the same sex?

**Example 5** The Alpha Beta Zeta sorority is trying to fill a pledge class of nine new members during fall rush. Among the twenty-five available candidates, fifteen have been judged marginally acceptable and ten highly desirable. How many ways can the pledge class be chosen to give a two-to-one ratio of highly desirable to marginally acceptable candidates?

**Example 6** Prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

**Example 7** Show that if $n$ is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

**Example 8** What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

**Example 9** Give a formula for the coefficient of $x^k$ in the expansion of $(x^2 - 1/x)^{100}$, where $k$ is an integer.

**Example 10** Find the expansion of $(x+y)^6$ using the binomial theorem.

### The Principle of Inclusion - Exclusion

Let $A_1, A_2, A_3, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

When $n = 2$, we have $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

When $n = 3$, we have $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.

**Example 11** Give a formula for the number of elements in the union of four sets.

**Example 12** How many positive integers not exceeding 1000 are divisible by 7 or 11?