Tangent Planes and Linear Approximations

1. The volume $V = V(p, T)$ of a specific quantity of a gas is a function of the pressure p and the temperature T. Suppose that V is measured in cubic feet, T is in °F, and p is in lb/in^2. Suppose, further, that $V(24, 500) = 23.69$.

   a) Thinking physically about the situation, should $\partial V/\partial p$ at $(24, 500)$ be positive or negative? Explain briefly.

   b) Suppose that you can reliably measure V when p changes by as small an increment as $2\,\text{lb/in}^2$ and/or when T changes by as small an increment as $20\,°F$.

   If you're going to take a measurement of V at just one new point (p_1, T_1), where $p_1 \geq 24$ and $T_1 \geq 500$, what should you pick for (p_1, T_1) in order to have the data that you need to obtain a good approximation of $\partial V/\partial p$ at $(24, 500)$? (You are **not** being asked to produce the approximation in this part of the problem; you are just supposed to supply the point (p_1, T_1)).

   c) Assume that $V(p_1, T_1) = 21.86$, where (p_1, T_1) is the point that you supplied above. What approximation do you obtain for $\partial V/\partial p$ at $(p, T) = (24, 500)$?

   d) Assume that $\partial V/\partial T = 0.0255\,\text{ft}^3/°\text{F}$ at $(p, T) = (24, 500)$. Combining this data with the data above, what do you obtain for the linearization of the function V at $(p, T) = (24, 500)$?

2. Consider the function $f(x, y, z) = xe^y + xz^2$.

(a) Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of the function f at the point $\mathbf{p} = (2, 0, 1)$.

(b) Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at $\mathbf{p}$.

(c) Use the linearization of f to estimate the value of f at $(1.9, 0.1, 1.5)$. (The exact value of $f(1.9, 0.1, 1.5)$ from your calculator is worth zero points.)

3. If $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$, estimate $f(1.01, 0.98)$.

4. The wind-chill index $W = W(T, v)$ is the perceived temperature when the actual temperature is T and the wind speed is v. Suppose that W and T are measured in °C and v is measured in km/h. Assume that $W(-20, 50) = -35$ and $W(-20.5, 50) = -35.6$.

a) Use data to estimate $\partial W/\partial T$ at the point $(T, v) = (-20, 50)$.

b) Suppose that the value of $\partial W/\partial v$ at $(-20, 50)$ is -0.2 in °C/(km/h). Combining this with the data above, what do you obtain for the linearization of the function W at $(T, v) = (-20, 50)$?

5. The radius of a spherical hot air balloon was estimated to be 4 meters with a possible error of at most 0.5 meters. What is the maximum error you can make in calculating the surface area of the balloon using the estimate of 4 meters?