Please do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

MATH 241 Fall 2022 Final Exam

December 15, 2022

Your name____

The exam has 12 different printed sides of exam problems, 1 side workspace and 1 side formula sheet.

Duration of the Final Exam is two and a half hours. There are 12 problems, 10 points each. Only 10 problems will be graded. If you solve more than 10 problems, you must cross out the problem(s) in the box above that must not be graded. If you solve more than 10 problems and do not cross out problems, only the first ten problems will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1) (5 points each)

(i) The plane

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{3} = 1$$

intersects the x-, y-, and z-axes in points P, Q, and R. Find the area of the triangle $\triangle PQR$.

(ii) Compute the volume of the parallelepiped spanned by

$$\mathbf{a} = \langle 1,0,0\rangle, \quad \mathbf{b} = \langle 0,2,0\rangle, \quad \mathbf{c} = \langle 1,1,2\rangle$$

2) (5 points each)

(i) Compute the cosine of the angle between the two planes

$$2x + 3y + 7z = 2 \quad \text{and} \quad 4x - 2y + 2z = 4$$

(ii) Find a vector equation, parametric equations and symmetric equations for the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

3)

(i) (6 points) Find an arc length parametrization of the circle in the plane $z = 9$ with radius 4 and center $(1, 4, 9)$.

(ii) (1 point) What is the unit tangent vector of a line with direction vector $\mathbf{v} = \langle 2, 1, -2 \rangle$?

(iii) (1 point) What is the curvature of a circle of radius 4?

(iv) (1 point) Which has larger curvature, a circle of radius 2 or a circle of radius 4?

(v) (1 point) What is the curvature of $\mathbf{r}(t) = \langle 2 + 3t, 7t, 5 - t \rangle$?

4)

(i) (3 points) Compute f_{uvxyvu} for

$$f(x, y, u, v) = \frac{x^2 + e^y v}{3y^2 + \ln(2 + u^2)}$$

(ii) (7 points) Use the linear approximation to estimate

$$(3.99)^3(1.01)^4(1.98)^{-1}$$

5) (5 points each)

(i) Find an equation of the tangent plane to the surface at the given point.

$$xz + 2x^2y + y^2z^3 = 11, \quad P = (2, 1, 1)$$

(ii) Find the point on the curve $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t \rangle$, $0 \le t \le \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

6) (5 points each)

(i) Suppose that a heated metal plate occupies the portion of the xy-plane where $x^2 + y^2 \leq 25$, where x and y are in meters. The plate is heated in such a way that, at the point $(x, y) = (-2\sqrt{2}, 2\sqrt{2})$, the partial derivatives of the temperature, T , in degrees Celsius, are given by

$$\frac{\partial T}{\partial x} = 3^\circ \text{ C/m} \quad \text{and} \quad \frac{\partial T}{\partial y} = -1^\circ \text{ C/m}.$$

Recall that, in polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. Calculate $\partial T/\partial r$ and $\partial T/\partial\theta$ at $(r, \theta) = (4, 3\pi/4)$, where r is in meters and θ is in radius. Physically, what do $\partial T/\partial r$ and $\partial T/\partial\theta$ measure?

(ii) Suppose that the xy-plane has an electric charge, in coulombs, given by

$$Q = xy + x^2\sin(\pi y),$$

where x and y are in meters. Also, suppose that a particle is at the point $(1, 1)$. If the particle moves in the direction in which the charge increases most rapidly, in what direction does it move? If the particle moves in the direction in which the charge increases most rapidly, at what rate is the charge changing, in coulombs per meters, when the particle is at $(1, 1)$?

7) (10 points) Find the critical points of the following function, and classify each one as a point where f has a local maximum value, a local minimum value, or a saddle point.

$$f(x, y) = x^3 - 3x + 3xy^2$$

8) (10 points) Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

9) (5 points each) Evaluate the iterated integrals.

(i) $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$

(ii) Calculate the surface area of the portion of the elliptic paraboloid $z = 4 - x^2 - y^2$, which is inside the cylinder $x^2 + y^2 = 1$.

10) (10 points) Let S be the solid region in the 1st octant (i.e., where $x \geq 0$, $y \geq 0$, and $z \geq 0$) in $\mathbb{R}^3$ which lies inside the (top) hemisphere where $x^2 + y^2 + z^2 = 4$ and $z \geq 0$ but lies outside the cylinder where $x^2 + y^2 = 1$, and bounded by the planes with equations $y = x$ and $y\sqrt{3} = x$. Assume that x, y, and z are measured in meters. Suppose the density of the solid region S is given by $\delta(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$ kg/m^3. Find the total mass of the solid region S.

11) Consider the vector field $\mathbf{F} = \langle e^y, xe^y, 2e^{2z} \rangle$.

a) (7 points) Determine if the vector field $\mathbf{F}$ is a conservative vector field. If $\mathbf{F}$ is conservative, find a potential function for $\mathbf{F}$.

b) (3 points) Let C be the curve parameterized by $\mathbf{r}(t) = \langle t^4, t, t^3 \rangle$, where $-1 \le t \le 1$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field given in part (a).

12) (5 points each)

(i) Compute the line integral of $\mathbf{F} = \langle e^{x+y}, e^{x-y} \rangle$ along the curve (oriented clockwise) consisting of the line segments by joining the points $(0,0), (2,2), (4,2), (2,0)$, and back to $(0,0)$.

(ii) Compute the area of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

using a line integral.

Workspace

Trigonometric Identities

$\sin^2\theta + \cos^2\theta = 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

Polar Coordinates

$x = r\,\cos\theta,\ y = r\,\sin\theta,\ r^2 = x^2 + y^2,\quad dA = r\,dr\,d\theta,\quad \tan\theta = \dfrac{y}{x}$

Cylindrical Coordinates

$x = r\,\cos\theta,\ y = r\,\sin\theta,\ z = z,\ r^2 = x^2 + y^2,\quad dV = r\,dz\,dr\,d\theta,\quad \tan\theta = \dfrac{y}{x}$

Spherical Coordinates

$x = \rho\,\sin\phi\,cos\theta,\quad y = \rho\,\sin\phi\ sin\theta,\ z = \rho\,\cos\phi$

$dV = \rho^2\,\sin\phi\,d\rho\,d\phi\,d\theta,\ \ \rho^2 = x^2 + y^2 + z^2,\ \ \tan\theta = \dfrac{y}{x}$

where $\rho \geq 0,\ 0 \leq \theta < 2\pi,$ and $0 \leq \phi \leq \pi$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} \ = \ \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} \ = \ f(b) - f(a)$$

$$\int_{\partial R} \mathbf{F}\cdot d\mathbf{r} \ = \ \int\int_R (Q_x - P_y)\ dA$$

$$\int\int_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA \quad \text{or} \quad \int\int_D \sqrt{f_x^2 + f_y^2 + 1}\ dA$$

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|},\quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},\quad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$$