1. Prove that if a permutation α can be expressed as a product of an even (odd) number of 2-cycles, then every decomposition of α into a product of 2-cycles must have an even (odd) number of 2-cycles.

2. Let $3 \leq m \leq n$. Calculate $\sigma\tau^{-1}$ for the cycles $\sigma = (1, 2, \ldots, m-1)$ and $\tau = (1, 2, \ldots, m-1, m)$ in S_n.

3. Prove that in S_n, with $n \geq 3$, any even permutation is a product of cycles of length three.
   Hint: $(a, b)(b, c) = (a, b, c)$ and $(a, b)(c, d) = (a, b, c)(b, c, d)$.

4. Prove that (a, b) cannot be written as a product of two cycles of length three.

5. Let S be any nonempty set, and let $\sigma \in \mathrm{Sym}(S)$. For $x, y \in S$ define $x \sim y$ if $\sigma^n(x) = y$ for some $n \in \mathbb{Z}$. Show that $\sim$ defines an equivalence relation on S.

6. Find $\langle \pi \rangle$ in $\mathbb{R}^\times$.

7. Prove that $\displaystyle\sum_{d|n} \phi(d) = n$ for any positive integer n.
   Hint: Interpret the equation in the cyclic group $\mathbb{Z}_n$, by considering all of its subgroups.

8. Let $n = 2^k$ for $k > 2$. Prove that $\mathbb{Z}_n^\times$ is not cyclic.
   Hint: Show that ± 1 and $(n/2) \pm 1$ satisfy the equation $x^2 = 1$, and that this is impossible in any cyclic group.

9. Show that no proper subgroup of S_4 contains both $(1, 2, 3, 4)$ and $(1, 2)$.

10. Show that the following matrices form a subgroup of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to D_4:
    $\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pm\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pm\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$

11. (a) Show that $A_4 = \{\sigma \in S_4 \mid s = \tau^2 \text{ for some } \tau \in S_4\}$.
    (b) Show that $A_5 = \{\sigma \in S_5 \mid s = \tau^2 \text{ for some } \tau \in S_5\}$.
    (c) Show that $A_6 \supset \{\sigma \in S_6 \mid s = \tau^2 \text{ for some } \tau \in S_6\}$.
    (d) What can you say about A_n if $n > 6$?

12. Show that in S_n the only elements which commute with the cycle $(1, 2, \ldots, n)$ are its powers.

13. Show that the product of two transpositions is one of (i) the identity; (ii) a 3-cycle; (iii) a product of two (nondisjoint) 3-cycles. Deduce that every element of A_n can be written as a product of 3-cycles.

14. Prove that every group of order n is isomorphic to a subgroup of $\mathrm{GL_n}(\mathbb{R})$.

15. Show that multiplicative group $\mathbb{Z}_{10}^\times$ is isomorphic to the additive group $\mathbb{Z}_4$.
    Hint: Find a generator a of $\mathbb{Z}_{10}^\times$ and define $\phi : \mathbb{Z}_4 \to \mathbb{Z}_{10}^\times$ by $\phi(n \pmod 4) = a^n \pmod{10}$.

16. Show that multiplicative group $\mathbb{Z}_7^\times$ is isomorphic to the additive group $\mathbb{Z}_6$.

17. Show that $\mathbb{Z}_5^\times$ is not isomorphic to $\mathbb{Z}_8^\times$ by showing that the first group has an element of order 4 but the second group does not.

18. Is the additive group $\mathbb{C}$ of complex numbers isomorphic to the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers?

19. Prove that any group with three elements must be isomorphic to $\mathbb{Z}_3$.

20. Let G be the following set of matrices of $\mathbb{R}$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$
    Show that G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

21. Let G be any group, and let a be a fixed element of G. Define a function $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$, for all $x \in G$. Show that ϕ_a is an isomorphism.

22. Let G be any group. Define $\phi : G \to G$ by $\phi(x) = x^{-1}$, for all $x \in G$.
    (a) Prove that ϕ is one-to-one and onto.
    (b) Prove that ϕ is an isomorphism if and only if G is abelian.

23. Define $\phi : \mathbb{C}^\times \to \mathbb{C}^\times$ by $\phi(a+bi) = a-bi$, for all nonzero complex numbers $a+bi$. Show that ϕ is an isomorphism.

24. Prove that if m, n are positive integers such that $\gcd(m,n) = 1$, then $\mathbb{Z}_{mn}^\times$ is isomorphic to $\mathbb{Z}_m^\times \times \mathbb{Z}_n^\times$.

25. Show that the cyclic group $\mathbb{Z}_4$ and the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic. Also, find two abelian groups of order 8 that are not isomorphic.