Your Name:

Duration of the mid-term exam is 90 minutes. There are five problems, worth 50 points, and an extra credit problem, worth 2 points. Show all your work for full credit. Books, notes etc. are prohibited. Calculators are NOT permitted.

1. (a) (5 points) In $\mathrm{GL}_2(\mathbb{R})$, compute the centralizer of the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

   (b) (5 points) Compute the center of $\mathrm{GL}_2(\mathbb{R})$.

2. (a) (5 points) Find the cyclic subgroup of $\mathbb{C}^\times$ generated by $(-\sqrt{2}+\sqrt{2}i)/2$.

(b) (5 points) In $\mathrm{GL}_2(\mathbb{R})$, find the order of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

3. (a) (5 points) Let $G = \mathrm{GL}_2(\mathbb{R})$. Does the subset S of G defined by

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| b = c \right\}$$

of symmetric 2×2 matrices form a subgroup of G?

(b) (5 points) Let $G = \langle a \rangle$ and $|a| = 20$. Find all the subgroups of G.

4. (5 points each) Let a be an element of the group G. Then prove the following.

(i) If a has infinite order, then $a^k \neq a^m$ for all integers $k \neq m$.

(ii) If a has finite order and $k \in \mathbb{Z}$, then $a^k = e$ if and only if $|a|$ divides k.

(iii) If a has finite order $|a| = n$, then for all integers k, m, we have $a^k = a^m$ if and only if $k \equiv m \pmod{n}$. Furthermore, $|\langle a \rangle| = |a|$.

5. (5 points) State Euler's Theorem and give a group theoretic proof of it.

EXTRA CREDIT PROBLEM Let G be a finite group, let $n > 2$ be an integer, and let S be the set of elements of G that have order n. Show that S has an even number of elements.

WORKSHEET