MATH 351 - 01 **Modern Algebra** **Homework Assignment 3**

Instructions

1. There are five problems, worth 50 points. You must submit solutions by 12:05pm on February 21, 2022.
2. You may refer to lecture notes, the textbook, video lectures and lecture notes posted on Canvas.
3. You may not consult any external resources. This means no internet searches, materials from other classes or books or any notes you have taken in other classes. You may not use Google or any other search engines for any reason. You may not use any shared Google documents.
4. You may not discuss this homework assignment or questions related to the assignment with any person outside class.
5. Show ALL your work for full credit.
6. You must sign the front page of the homework assignment and submit it with your solutions. If you do not submit the signed first page, your solutions will NOT be graded and it will be worth ZERO points.
7. Please use A4 sheets to write your solutions and start each problem on a new sheet of paper. Please make sure your solutions are very clear and legible.
8. Please follow the guidelines given in the syllabus on how to prepare and how to submit your homework assignment.

All work submitted is mine and mine alone.

I have read and followed the instructions above.

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Signature

1. (5 points each) For each of the following relations on $\mathbb{R}$, determine which of the three conditions of an equivalence relation hold.

   (a) For $a, b \in \mathbb{R}$, define $a \sim b$ if $a - b \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers.

   (b) For $a, b \in \mathbb{R}$, define $a \sim b$ if $|a - b| \leq 1$.

2. (a) (3 points) Define an equivalence relation on the set $\mathbb{R}$ that partitions the real line into subsets of length 1.

   (b) (3 points) In $\mathbb{R}^3$, consider the standard (x, y, z)-coordinate system. We can define a partition of $\mathbb{R}^3$ by using planes parallel to the (x, y)-plane. Describe the corresponding equivalence relation by giving conditions on the coordinates x, y, z.

3. (10 points) For integers m, n, define $m \sim n$ if and only if $n | m^k$ and $m | n^j$ for some positive integers k and j.

   (i) Show that $\sim$ is an equivalence relation on $\mathbb{Z}$.

   (ii) Determine the equivalence classes $[1], [2], [6]$ and $[12]$.

   (iii) Give a characterization of the equivalence class $[m]$.

4. (a) (5 points) Let S be a set. A subset $R \subseteq S \times S$ is called a **circular relation** if (i) for each $a \in S$, $(a, a) \in R$ and (ii) for each $a, b, c \in S$, if $(a, b) \in R$ and $(b, c) \in R$, then $(c, a) \in R$. Show that any circular relation must be an equivalence relation.

   (b) (5 points) Let W be a subspace of a vector space V over $\mathbb{R}$, (that is, the scalars are assumed to be real numbers). We say that two vectors $\mathbf{u}, \mathbf{v} \in V$ are congruent modulo W if $\mathbf{u} - \mathbf{v} \in W$, written $\mathbf{u} \equiv \mathbf{v} \pmod{W}$. Show that $\equiv$ is an equivalence relation.

5. Let S be a set and let $2^S = \{A | A \subseteq S\}$ be the collection of all subsets of S. Define $\sim$ on 2^S by letting $A \sim B$ if and only if there exists a one-to-one correspondence from A to B.

   (a) (6 points) Show that $\sim$ is an equivalence relation on 2^S.

   (b) (8 points) If $S = \{1, 2, 3, 4\}$, list the elements of 2^S and find each equivalence class determined by $\sim$.