COLLEGE OF THE HOLY CROSS
Department of Mathematics and Computer Science

Academic Year 2021/2022 - Spring 2022
MATH 351 (01) - Modern Algebra
Final Exam

1	2	3	4	5	6	7	8	9	10	11	12	13	14	total

Your name____

Duration of the Final Exam is two and a half hours. There are 14 problems, 10 points each. Only 12 problems will be graded. If you solve more than 12 problems, you must cross out the problem(s) in the box above that must not be graded. If you solve more than 12 problems and do not cross out problems, only the first twelve problems will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

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1. Which of the following are equivalence relations on the set of integers? For the equivalence relations, describe the distinct equivalence classes, and for the non-equivalence relations, explain which of the three properties of an equivalence relation fail.

   (i) $a \sim b$ if $a - b$ is a multiple of 5.

   (ii) $a \sim b$ if $a + b$ is a multiple of 5.

2. Prove or disprove that each of the following subsets of $\mathrm{GL}_2(\mathbb{R})$ is a group. In the case of non-groups, indicate which of the group conditions fail.

(i) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; ad - bc = 2 \right\}$

(ii) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; ad - bc \in \{-1, 1\} \right\}$

(iii) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; c = 0 \right\}$

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(iv) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; d = 0 \right\}$

(v) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; a = d = 1 \text{ and } c = 0 \right\}$

3. (a) Let $G = \langle a \rangle$ and $|a| = 64$. Find all the subgroups of G.

(b) Find all elements of G which generate G.

(c) Let H be the unique subgroup of G of order 16. Find all elements of order 8 in H.

4. (a) Find the order of the cyclic subgroup of $\mathrm{GL}_2(\mathbb{R})$ generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) Find the cyclic subgroup of $\mathbb{C}^\times$ generated by $(-1+\sqrt{3}i)/2$.

5. Consider the element $\alpha = (5, 4, 1)(3, 7, 4, 2)(1, 2, 6, 5) \in S_7$.

   (a) Write α in disjoint cycle form.

   (b) Find the order of α.

   (c) If $\alpha \in A_7$?

   (d) Find α^{-1}, expressing it also in disjoint cycle form.

6. (a) Find all possible orders of elements in S_5.

(b) Find the number of permutations of each possible order in S_5.

7. Prove or disprove that D_{12} and S_4 are isomorphic.

Hint:
i) List the elements of D_{12} and find the order of each element.
ii) For $a, b \in D_n$ with $|a| = n$ and $|b| = 2$, we have $ba^m = a^{-m}b$, for all $m \in \mathbb{Z}$.
iii) Find all possible orders of elements of S_4.

8. (a) Consider the subgroup $\langle i \rangle = \{\pm 1, \pm i\}$ of $\mathbb{C}^{\times}$. Write down the Cayley table for $\langle i \rangle$.

(b) Write down the Cayley table for $\mathbb{Z}_4$.

(c) Show that $\mathbb{Z}_4$ is isomorphic to $\langle i \rangle$.

Note: You must define a map from $\mathbb{Z}_4$ to $\langle i \rangle$, and then show that it is well-defined, 1-1, onto and operation-preserving.

9. (a) In S_3, compute the centralizer of $\sigma = (1, 2, 3)$.

(b) In S_3, compute the centralizer of $\sigma = (1, 2)$.

10. Prove that the set of all 5-th roots of unity is a group of order 5 under the usual multiplication of complex numbers.

    Note: You must find all the roots of $z^5 = 1$ and then prove that the set of all roots satisfies the axioms of a group. Also, identify the inverse of each element. You may use the fact that the multiplication of complex numbers is associative.

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11. Let G be a finite group of order n. Prove the following.

    (i) For any $a \in G$, $|a|$ is a divisor of n.

    (ii) For any $a \in G$, $a^n = e$.

12. Prove that the set of even permutations in S_n forms a subgroup of S_n.

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13. Let S be any set. Prove that if σ and τ are disjoint cycles in $\mathrm{Sym}(S)$, then $\sigma\tau = \tau\sigma$.

14. Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Then prove the following.

    (i) ϕ carries the identity of G to the identity of $\overline{G}$.

    (ii) For every integer n and for every group element $a \in G$, $\phi(a^n) = [\phi(a)]^n$.

    Hint: You must prove three cases: $n \in \mathbb{Z}^+$, $n = 0$ and $n \in \mathbb{Z}^-$.

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(iii) $|a| = |\phi(a)|$ for all a in G. (isomorphisms preserve orders).

(iv) Let $a, b \in G$. Then a and b commute if and only if $\phi(a)$ and $\phi(b)$ commute.

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WORKSHEET

End of exam