(1) Consider the set $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$. We usually denote this set of functions by $C(\mathbb{R})$. Recall that f is continuous if and only if for all $a \in \mathbb{R}$, we have $\lim_{x \to a} f(x) = f(a)$. We define vector sum and scalar multiplication in $C(\mathbb{R})$ as usual for functions. Prove that $C(\mathbb{R})$ is a vector space.

(2) Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable everywhere, and } f'(x) \in C(\mathbb{R})\}$. The elements of V are called continuously differentiable functions and the set V is usually denoted by $C^1(\mathbb{R})$. Show that $C^1(\mathbb{R})$ is a subspace of $C(\mathbb{R})$.

(3) Let $C^2(\mathbb{R})$ denote the set of functions $f \in C(\mathbb{R})$ such that f is twice differentiable and $f'' \in C(\mathbb{R})$. Show that $C^2(\mathbb{R})$ is a subspace of $C(\mathbb{R})$.

(4) Determine whether W is a subspace of V.

(a) $V = \mathbb{R}^3$, $W = \{(x_1, x_2, x_3) \mid 4x_1 + 3x_2 - 2x_3 = 0 \text{ and } x_1 - x_3 = 0\}$.

(b) $V = C(\mathbb{R})$, $W = \{f \in C(\mathbb{R}) \mid f(2) = 0\}$.

(c) $V = \mathbb{R}^2$, $W = \{(x_1, x_2) \mid x_1^3 - x_2^2 = 0\}$.

(d) $V = C^1(\mathbb{R}),\ \ W = \{f \mid f'(x) + 4f(x) = 0 \text{ for all } x \in \mathbb{R}\}$.

(e) $V = C^1(\mathbb{R}),\ \ W = \{f \mid \sin(x) \cdot f'(x) + f(x) = 6 \text{ for all } x \in \mathbb{R}\}$.

(5) Let W be a subspace of a vector space V, let $\mathbf{y} \in V$, and define the set

$$\mathbf{y} + W = \{\mathbf{x} \in V \mid \mathbf{x} = \mathbf{y} + \mathbf{w} \text{ for some } \mathbf{w} \in W\}.$$

Show that $\mathbf{y} + W$ is a subspace of V if and only if $\mathbf{y} \in W$.

(6) (a) Let $F([a, b])$ denote the set of all functions $f : [a, b] \to \mathbb{R}$, and $C([a, b])$ denote the set of continuous functions on the closed interval $[a, b] \subset \mathbb{R}$. Show that $C([a, b])$ is a subspace of the vector space $F([a, b])$.

(b) Let $C^\infty(\mathbb{R})$ denote the set of functions in $F(\mathbb{R})$ that have derivatives of all orders. Show that $C^\infty(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

(7) (a) Show that the only subspaces of $\mathbb{R}^2$ are the zero subspace, $\mathbb{R}^2$ itself, and the lines through the origin.

(b) Show that if V_1 is a subspace of V_2 and V_2 is a subspace of V_3, then V_1 is a subspace of V_3.

(8) Show that the subset $W = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}(\mathbb{R})] \,\middle|\, 3a_{11} - 2a_{22} = 0 \right\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

(9) Show that the subset $W = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}(\mathbb{R})] \,\middle|\, a_{12} = a_{21} \right\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$. The set W is the set of symmetric matrices of size 2×2.

(10) Show that the subset $W = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}(\mathbb{R})] \,\middle|\, a_{11} = a_{22} = 0 \text{ and } a_{12} = -a_{21} \right\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$. The set W is the set of skew-symmetric matrices of size 2×2.