Due by 4pm on Friday, April 11. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code.

(1) (4 points each) Determine whether the following functions $T : V \to W$ define linear transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T(x_1, x_2) = (x_2, x_1, x_1, x_2)$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (1 + x_1, 2x_2, x_3 - x_2)$.

(c) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x_1, x_2, x_3) = 3x_1 + 2x_2 + x_3$.

(d) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x_1, x_2, x_3) = e^{x_1+x_2+x_3}$.

(e) $\mathrm{Int} : C[a, b] \to \mathbb{R}$ defined by $\mathrm{Int}(f) = \int_a^b f(x)\,dx$.

(2) (5 points each)

(a) If $T : \mathbb{R}^2 \to \mathcal{P}_3(\mathbb{R})$ is a linear transformation satisfying $T(1, 1) = x + x^2$ and $T(3, 0) = x - x^3$, what is $T(2, 2)$?

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation satisfying $T(1, 0, 1) = (1, 0)$, $T(0, 1, 1) = (0, 1)$, and $T(1, 1, 0) = (1, 1)$, what is $T(2, -4, 3)$?

(3) (5 points each)

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1 + x_2 - x_3, x_3 - x_1).$$

What is the matrix of T with respect to the standard bases in V and W?

(b) Let V be the subspace of $C(\mathbb{R})$ spanned by $\sin x$ and $\cos x$. Define $D : V \to V$ by $D(f(x)) = f'(x)$. What is the matrix of D with respect to the basis $\{\sin x, \cos x\}$?

(c) Let $\mathbf{a} = (a_1, \dots, a_n)$ be a fixed vector in $\mathbb{R}^n$. Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(\mathbf{v}) = \sum_{i=1}^{n} a_i v_i$. What is the matrix of T with respect to the standard bases in V and W?

(4) (5 points each) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 - x_2 + x_3).$$

(a) Let β be the standard basis in $\mathbb{R}^3$. Find $[T]_\beta^\beta$.

(b) Let $U \subset \mathbb{R}^3$ be the subspace spanned by $\alpha = \{(1, 1, 0), (0, 0, 1)\}$, and let $S : U \to \mathbb{R}^3$ be the restriction of T to U: that is, $S = T|_U$. Compute $[S]_\alpha^\beta$.

(c) Is there any relationship between your answers to part (a) and part (b)?

(5) (10 points each) For each of the following matrices, defining linear maps T between vector spaces of the appropriate dimensions, find bases for $\mathrm{Ket}(T)$ and $\mathrm{Im}(T)$.

(a) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -4 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 3 \\ 1 & 2 & 5 & 6 \end{bmatrix}$

(6) (5 points each)

(a) Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by $T(p(x)) = xp(x)$. Is T injective, surjective, both, or neither?

(b) Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by $T(p(x)) = \frac{d}{dx}p(x)$. Is T injective, surjective, both, or neither?

In each of the following two cases, determine whether $T : \mathbb{R}^k \to \mathbb{R}^l$ is injective, surjective, both, or neither, where T is defined by the matrix.

(c) $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$