Due by 4pm on Friday, April 4. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code. Each problem is worth 20 points.

Please read the following note before you begin the homework assignment.

Let W_1 and W_2 be subspaces of a vector space V. In class, we discussed the sum $W_1 + W_2$ defined as follows:

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 \,:\, \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

$W_1 + W_2$ is a subspace of V. In fact, $W_1 + W_2$ is the smallest subspace of V containing W_1 and W_2. This means, any subspace that contains W_1 and W_2 also contains $W_1 + W_2$.

The sum $W_1 + W_2$ is called a *direct sum* if each element of $W_1 + W_2$ can be written in only one way as a sum $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$. If $W_1 + W_2$ is a direct sum, we write the sum as $W_1 \oplus W_2$.

Condition for a direct sum Suppose W_1 and W_2 are subspaces of V. Then $W_1 + W_2$ is a direct sum if and only if the only way to write $\mathbf{0}$ as a sum $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$, is by taking each $\mathbf{w}_k$ equal to $\mathbf{0}$.

Direct sum of two subspaces Suppose U and W are subspaces of V. Then

$$U + W \text{ is a direct sum} \Longleftrightarrow U \cap W = \{\mathbf{0}\}$$

(1) (10 points each) Prove the following:

(a) If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

(b) Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

(c) Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

(d) Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

(2) (10 points each)

(a) Let $U = \{(x, y, 0) \in \mathbb{R}^3 \,|\, x, y \in \mathbb{R}\}$ and $W = \{(0, 0, z) \in \mathbb{R}^3 \,|\, z \in \mathbb{R}\}$. Show that $\mathbb{R}^3 = U \oplus W$.

(b) Suppose

$$V_1 = \{(x, y, 0) \in \mathbb{R}^3 \,|\, x, y \in \mathbb{R}\}, \quad V_2 = \{(0, 0, z) \in \mathbb{R}^3 \,|\, z \in \mathbb{R}\}, \; V_3 = \{(0, y, y) \in \mathbb{R}^3 \,|\, y \in \mathbb{R}\}.$$

Show that $\mathbb{R}^3 = V_1 + V_2 + V_3$. Is it a direct sum, $\mathbb{R}^3 = V_1 \oplus V_2 \oplus V_3$? Justify your answer.

(3) (10 points each)

(a) Suppose

$$U = \{(x, x, y, y) \in \mathbb{R}^4 \,:\, x, y \in \mathbb{R}\}.$$

Find a subspace W of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W$.

(b) Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{R}^5 \,:\, x, y \in \mathbb{R}\}.$$

Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

(4) (10 points each)

(a) Let $\mathcal{U} = \{p \in \mathcal{P}_4(\mathbb{R}) \,:\, p(6) = 0\}$. You found a basis of U in Worksheet 8. Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

(b) Let $\mathcal{U} = \{p \in \mathcal{P}_4(\mathbb{R}) \,:\, p''(6) = 0\}$. You found a basis of U in Homework 6. Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.