Due by 4pm on Friday, March 28. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code. There are five problems, 20 points each.

(1) (a) Let $\mathcal{U} = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$.

(i) Show that $\mathcal{U}$ is a subspace of $\mathcal{P}_4(\mathbb{R})$.

(ii) Find a basis of $\mathcal{U}$.

(iii) Extend the basis in (b) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(b) Let $\mathcal{U} = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$.

(i) Show that $\mathcal{U}$ is a subspace of $\mathcal{P}_4(\mathbb{R})$.

(ii) Find a basis of $\mathcal{U}$.

(iii) Extend the basis in (b) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(2) (a) Show that there is a unique basis $\{p_1, p_2, p_3\}$ of $P_2(\mathbb{R})$ with the property that $p_1(0) = 1, p_1(1) = p_1(2) = 0$, $p_2(1) = 1$, $p_2(0) = p_2(2) = 0$, and $p_3(2) = 1$, $p_3(0) = p_3(1) = 0$.

(b) Find a basis for and the dimension of the subspace W of $\mathcal{P}_4(\mathbb{R})$ defined by

$$W = \{p \in \mathcal{P}_4(\mathbb{R} \,|\, P(1) = p(-1) = 0\}.$$

(3) (a) Show that $S = \{(1,1,1), (1,1,0), (1,0,0)\}$ and $S' = \{(1,1,1), (0,1,1), (0,0,1)\}$ are both bases of $\mathbb{R}^3$.

(b) Express each vector in S as a linear combination of the vectors in S'.

(4) (a) Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbb{R})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbb{R})$.

(b) Prove or give a counterexample: If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ is a basis of V and U is a subspace of V such that $\mathbf{v}_1, \mathbf{v}_2 \in U$ and $\mathbf{v}_3 \notin U$ and $\mathbf{v}_4 \notin U$, then $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis of U.

(5) (a) Find a basis for the kernel of the matrix $\begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & -6 & 2 & 5 \end{bmatrix}$. Justify your answer carefully; that is, explain how you know that the vectors you found are linearly independent and span the kernel.

(b) Find a basis of the image of the matrix given in part (a). Justify your answer carefully; that is, explain how you know that the vectors you found are linearly independent and span the image.

(c) Find dim Ker(A).

(d) Find dim Im(A).