Due by 4pm on Friday, March 14. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code. There are five problems, 20 points each.

(1) (a) Show that $\mathbb{R}^n$ with the usual vector addition and scalar multiplication is a vector space.

(b) Let $\mathcal{P}_n(\mathbb{R})$ be the set of all polynomials with real coefficients of degree at most n. Show that $P_n(\mathbb{R})$ is a vector space.

(2) In each of the following parts, decide if the set $\mathbb{R}^2$, with the given operations, is a vector space. If this is not the case, say which of the axioms fail to hold.

(a) vector sum $(x_1, x_2) + (y_1, y_2) = (x_1 + 2y_1, 3x_2 - y_2)$, and the usual scalar multiplication $c(x_1, x_2) = (cx_1, cx_2)$.

(b) usual vector sum $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$, and scalar multiplication $c(x_1, x_2) = (cx_1, (1/c)x_2)$ if $c \neq 0$ and $(0, 0)$ if $c = 0$.

(c) vector sum $(x_1, x_2) + (y_1, y_2) = (0, x_1 + y_2)$, and the usual scalar multiplication.

(d) vector sum $(x_1, x_2) + (y_1, y_2) = (x_1y_1, x_2y_2)$, and the usual scalar multiplication.

(3) (a) Let $F(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Show that $F(\mathbb{R})$ is a vector space under the usual operations $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for all $x \in \mathbb{R}$ where $c \in \mathbb{R}$.

(b) Is the set $F(\mathbb{R})$ a vector space if the sum operation is defined by $f + g = fg$ and scalar multiplication is defined by $c \cdot f = c + f$?

(c) Is the set $F(\mathbb{R})$ a vector space if the sum operation is defined by $f + g = f - g$ and scalar multiplication is defined by $(c \cdot f)(x) = f(cx)$?

(d) Is the set $F(\mathbb{R})$ a vector space if the sum operation is defined by $f + g = f \circ g$ (composition of functions), and usual scalar multiplication.

(4) (a) Show that in any vector space V

(i) If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, then $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$ implies $\mathbf{y} = \mathbf{z}$.

(ii) If $\mathbf{x}, \mathbf{y} \in V$ and $a, b \in \mathbb{R}$, then $(a + b)(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + b\mathbf{x} + a\mathbf{y} + b\mathbf{y}$.

(b) (i) What vector space might be used to describe (simultaneously) the position, velocity, and acceleration of an object moving along a path in the plane $\mathbb{R}^2$?

(ii) What vector space might be used to describe (simultaneously) the position, velocity, and acceleration of an object moving along a path in the plane $\mathbb{R}^3$?

(5) For each of the following subsets W of a vector space V, determine if W is a subspace of V. Say why or why not in each case:

(a) $V = \mathbb{R}^3$, and $W = \{(a_1, a_2, a_3) \,|\, a_1 - 3a_2 + 4a_3 = 0, \text{ and } a_1 = a_2\}$.

(b) $V = \mathbb{R}^2$, and $W = \{(a_1, a_2) \,|\, \sin(a_1) = a_2\}$.

(c) $V = \mathbb{R}^3$, and $W = \{(a_1, a_2, a_3) \,|\, (a_1 + a_2 + a_3)^2 = 0\}$.

(d) $V = \mathbb{R}^3$, and $W = \{(a_1, a_2, a_3) \,|\, a_3 \geq 0\}$.

(e) $V = \mathbb{R}^3$, and $W = \{(a_1, a_2, a_3) \,|\, a_1, a_2, a_3 \text{ all integers}\}$.

(f) $V = \mathcal{P}_n(\mathbb{R})$, and $W = \{p \,|\, p(\sqrt{2}) = 0\}$.

(g) $V = \mathcal{P}_n(\mathbb{R})$, and $W = \{p \,|\, p(1) = 1 \text{ and } p(2) = 0\}$.

(h) $V = \mathcal{P}_3(\mathbb{R})$, and $W = \{p \,|\, p'(x) \in \mathcal{P}_1(\mathbb{R})\}$.

(i) $V = F(\mathbb{R})$, and $W = \{f \,|\, f \text{ is periodic with period } 2\pi\}$.

(j) $V = R^{100\times100}$, and $W = \{A \in V \,|\, A \text{ is symmetric}\}$.