Due by 4pm on Friday, February 21. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code. There are five problems, worth 20 points each.

(1) A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following facts:

(a) All homogeneous systems are consistent.

(b) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

(c) If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_2$ is a solution as well.

(d) If $\vec{x}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and k is an arbitrary constant, then $k\vec{x}$ is a solution as well.

(2) Is the vector

$$\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}?$$

(3) Determine whether the statements that follow are true or false, and justify your answer.

(a) A system of four linear equations in three unknowns is always inconsistent.

(b) There exists a 3×4 matrix with rank 4.

(c) The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.

(d) A linear system with fewer unknowns than equations must have infinitely many solutions or none.

(e) If the 4×4 matrix A has rank 4, then any linear system with coefficient matrix A will have a unique solution.

(4) Give a geometrical description of the set of all vectors of the form

(a) $\vec{v} + c\vec{w}$, where c is an arbitrary real number.

(b) $\vec{v} + c\vec{w}$, where $0 \le c \le 1$.

(c) $a\vec{v} + b\vec{w}$, where $0 \le a \le 1$ and $0 \le b \le 1$.

(d) $a\vec{v} + b\vec{w}$, where $a + b = 1$.

(e) $a\vec{v} + b\vec{w}$, where $0 \le a$ and $0 \le b$, and $a + b \le 1$.

(5) (a) Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Use the Gauss-Jordan method to find A^{-1}.

(b) Let A and B be invertible $n \times n$ matrices. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

(c) Let A be an upper triangular matrix. Show that A is invertible if and only if the diagonal entires of A are all nonzero.

(d) Let A be an upper triangular matrix. Prove that A^{-1} is also an upper triangular matrix.

Note: In (c) and (d), the statements are still true if you replace "upper" by "lower".