Due by 4pm on Friday, February 14. Please leave your homework on the table before class begins on Friday or leave it in the dropbox outside my office. Do not forget to attach the honor code. Each problem is worth 20 points.

(1) (a) Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\9\\9\\7\end{bmatrix}.$$

(b) Write down the augmented matrix of the linear system that must be solved to find all solutions x_1, x_2, x_3 of the equation

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3,$$

where

$$\vec{b} = \begin{bmatrix}-8\\-1\\2\\15\end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix}1\\4\\7\\5\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}2\\5\\8\\3\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}4\\6\\9\\1\end{bmatrix}.$$

Use Mathematica to find the rref of the augmented matrix, and then find all solutions x_1, x_2, x_3. Attach a printout of your Mathematica code to your homework submission.

(2) Consider the equations

$$\begin{aligned} x + 2y + 3z &= 4 \\ x + ky + 4z &= 6 \\ x + 2y + (k+2)z &= 6, \end{aligned}$$

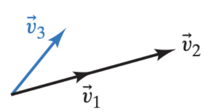
where k is an arbitrary constant.

(a) What is the augmented matrix of this system of equations?
(b) For which values of the constant k does this system have a unique solution?
(c) When is there no solution?
(d) When are there infinitely many solutions?

(3) (a) Consider the vectors $\vec{v}_1$, $\vec{v}_2$,, $\vec{v}_3$ in $\mathbb{R}^2$ shown in the accompanying figure. Vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

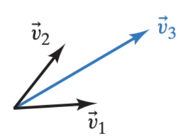
have? Argue geometrically.



(b) Consider the vectors $\vec{v}_1$, $\vec{v}_2$,, $\vec{v}_3$ in $\mathbb{R}^2$ shown in the accompanying figure. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

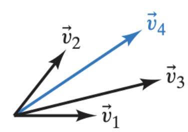
have? Argue geometrically.



(c) Consider the vectors $\vec{v}_1$, $\vec{v}_2$,, $\vec{v}_3$, $\vec{v}_4$ in $\mathbb{R}^2$ shown in the accompanying figure. Arguing geometrically, find two solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know that this system has, in fact, infinitely many solutions?



(4) (a) Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

(b) Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

(c) If the rank of a 4×4 matrix is 4, what is $\text{rref}(A)$?

(d) If the rank of a 5×3 matrix is 3, what is $\text{rref}(A)$?

(5) Let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find a *diagonal* matrix A such that $A\vec{x} = \vec{y}$.

(b) Find a matrix A of rank 1such that $A\vec{x} = \vec{y}$.

(c) Find an *upper triangular* matrix A such that $A\vec{x} = \vec{y}$, where all the entries of A on and above the diagonal are nonzero.