COLLEGE OF THE HOLY CROSS
Department of Mathematics and Computer Science

MATH 244 Linear Algebra
Spring 2025 Final Exam

Question	**1**	**2**	3	4	5	6	7	8	9	10	11	12	*EC*	Total
Points	10	10	10	10	10	10	10	10	10	10	10	10	10	100

Your name____

Duration of the Final Exam is 150 minutes. There are 12 problems. ***The first problem and the second problem are mandatory****. From problems 3 – 12, only 8 problems will be graded. If you solve all Problems 3 – 12, you must cross out the two problems in the boxes above that must not be graded. If you solve all Problems 3 – 12 but do not cross out two problems, only the first ten problems from 3 – 12 will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.*

1. Consider the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$.

   Determine whether the mapping is diagonalizable. If it is, find a basis (eigenbasis) of the appropriate vector space consisting of eigenvectors, and diagonalize A.

2. Determine whether the statements that follow are true or false. Give a reason or counterexample.

(a) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ are any two bases of a subspace V of $\mathbb{R}^{10}$, then n must equal m.

(b) If A is a 5×6 matrix of rank 4, then the nullity of A is 1.

(c) The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.

(d) There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$.

(e) If vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent, then vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must be linearly independent as well.

(f) Vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}$ form a basis of $\mathbb{R}^3$.

(g) Matrix $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ is similar to $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

(h) If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

(i) If a subspace V of $\mathbb{R}^3$ contains the standard vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, then V must be $\mathbb{R}^3$.

(j) There exists a 2×2 matrix A such that $\mathrm{im}(A) = \ker(A)$.

3. Write the matrix form of the following system, find the inverse of the coefficient matrix, and solve it.

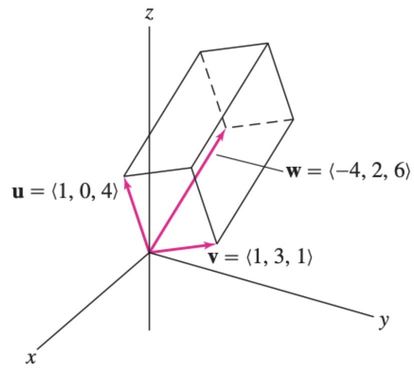
$$\begin{aligned} x + y + z &= 1 \\ x + 3y + 3z &= 5 \\ 2x + 2y + 5z &= 2 \end{aligned}$$

4. (a) For which values of the scalar λ is the matrix $\begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & 1 & \lambda \end{bmatrix}$ invertible?

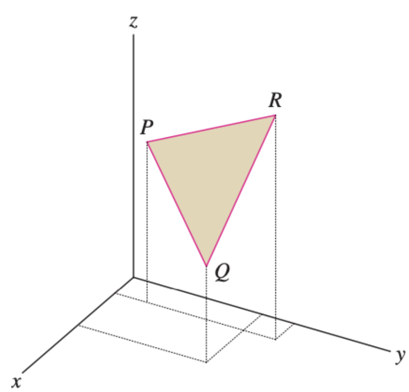
(b) Find a basis of the row space of the matrix. Also, find nullity(A). Mention any result/theorem used.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

5. (a) Find the volume of the parallelepiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$ in the following figure.



(b) Find the area of the triangle with vertices $P = (1, 1, 5)$, $Q = (3, 4, 3)$, and $R = (1, 5, 7)$.



6. Suppose

$$U = \{(x, y, x + y, x - y, 2x + y) \in \mathbb{R}^5 \; : \; x, y \in \mathbb{R}\}.$$

(a) Prove that U is a subspace of $\mathbb{R}^5$.

(b) Find a basis for U.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$. Show all your work.

7. (a) Determine if the given set of vectors is linearly independent. Mention any result/theorem used.

$$S = \{x^2 + x + 1,\ -x^2 + 2x,\ x^2 + 2,\ x^2 - x\} \text{ in } \mathcal{P}_2(\mathbb{R}).$$

(b) Is $\mathbf{v} = (1, 1, 0, 0) \in \operatorname{span}\big((1, 1, 1, 1), (1, -1, 1, 0), (0, 0, 0, 1)\big)$?

8. (a) Find a basis of $\mathcal{P}_5(\mathbb{R})$ containing the linearly independent set $S = \{x^5 - 2x,\ x^4 + 3x^2,\ 6x^5 + 2x^3\}$.

(b) Find a basis of $\mathbb{R}^3$ which is contained in the spanning set $S = \{(1, 2, 1),\ (-1, 3, 1),\ (0, 5, 2),\ (1, 1, 1),\ (0, 4, 2)\}$.

9. (a) Show that $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (2,3,1)$, $\mathbf{v}_3 = (4,1,5)$ is a basis of $\mathbb{R}^3$.

(b) Suppose that we define a linear transformation $T : \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ by setting

$$T(\mathbf{v}_1) = x^2 - 5, \;\; T(\mathbf{v}_2) = x^7 + 1, \;\; T(\mathbf{v}_3) = 11.$$

Find $T((1,2,5))$.

(c) Determine whether T is injective, surjective, both, or neither.

10. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(x, y, z) = (x + y + 2z,\ 2x + 2y + 4z,\ 2x + 3y + 5z).$$

Then T is a linear transformation.

(a) Let α be the standard basis of $\mathbb{R}^3$. Compute $[T]_\alpha^\alpha$.

(b) Find $\mathrm{Im}(T)$.

(c) Find a basis for $\mathrm{Im}(T)$.

(d) Find a basis for $\mathrm{Ker}(T)$.

(e) Determine whether T is injective, surjective, both, or neither.

11. (a) For which values of constants a, b, and c is the following matrix diagonalizable?

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Suppose there is an eigenbasis for a matrix A. What is the relationship between the algebraic multiplicities and dimensions of eigenspaces (geometric multiplicities) of its eigenvalues?

(c) Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Is there an eigenbasis? Interpret your result geometrically.

12. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$.

(a) Find an orthonormal basis for $\mathrm{Im}(A)$.

(b) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 5 \end{bmatrix}$. Find $\mathrm{proj}_{\mathrm{Im}(A)}(\mathbf{x})$.

(c) Find a basis for $\mathrm{Im}(A)^{\perp}$.

(d) Find a basis for $\mathrm{Ker}(A^T)$?

(e) Is $\mathrm{Im}(A)^{\perp} = \mathrm{Ker}(A^T)$?

EXTRA CREDIT PROBLEM

1. True/False.

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} = 1.$$

2. For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

describe the images and kernels of the matrices A, A^2 and A^3 geometrically.

WORKSHEET