Please do not write in the boxes immediately below.

problem	1	2	3	4	5	6	EC	total
points								

MATH 244 Linear Algebra Spring 2025 Exam 2

April 14, 2025

Your name____

The exam has 8 different printed sides of exam problems and 1 side workspace.

Duration of the exam is 90 minutes. There are 6 problems, worth 20 points each, and an extra credit problem, worth 5 points. From Problems 1 – 6, only 5 problems will be graded. If you solve all Problems 1 – 6, you must cross out the problem in the box above that must not be graded. If you solve all Problems 1 – 6 and do not cross out a problem, only the first five problems will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1. (a) Determine whether the following sets are real vector spaces. Justify your answer.

   i. $\{f \mid \text{the sum of coefficients of } f \text{ is } 0\}$

   ii. $\{f \mid f(3) = 2f(4)\}$

(b) Prove that the following set is a subspace of $\mathbb{R}^3$, and find a subspace W such that $R^3 = U \oplus W$.

$$U = \{(x_1, x_2, x_3) \,|\, 5x_1 + 3x_2 - 2x_3 = 0 \text{ and } x_1 - x_3 = 0\}.$$

2. Prove the following:

(a) If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

(b) Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

3. Consider the subset $S = \{x^3 - x, x^3 - x^2, x^3 - 1, 2x^2 - 3x + 1\}$ of $\mathcal{P}_3(\mathbb{R})$.

(a) Is S linearly independent? Justify your answer.

(b) Is $x - 1 \in \operatorname{span}(S)$? Justify your answer.

4. Let $\mathcal{U} = \{p \in \mathcal{P}_5(\mathbb{R}) \,:\, p''(6) = 0\}$.

(a) Show that $\mathcal{U}$ is a subspace of $\mathcal{P}_5(\mathbb{R})$

(b) Find a basis of $\mathcal{U}$.

(c) Extend it to a basis of $\mathcal{P}_5(\mathbb{R})$.

5. (a) Define $T : P(\mathbb{R}) \to P(\mathbb{R})$ by $T(p(x)) = x^2\, p'(x)$.

   i. Prove that T is a linear transformation.

   ii. Is T injective, surjective, both, or neither? Justify your answer.

   (b) True/False.

   i. A linear map from $P_4(\mathbb{R})$ to $P_3(\mathbb{R})$ is always injective.

   ii. A linear map from $\mathbb{R}^2$ to $\mathbb{R}^4$ is always surjective.

6. Let $T : V \to W$ be a linear transformation from a six-dimensional vector space V with basis $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ to a three-dimensional vector space W with basis $\beta = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ with matrix

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 2 & 0 \end{bmatrix}$$

Find a bases for $\ker(T)$ and bases for $\mathrm{Im}(T)$. Also, find a basis for V so that the first $\dim \mathrm{ket}(T)$ members are in $\ker(T)$.

EXTRA CREDIT PROBLEM Consider the matrices:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Describe each transformation in words.

WORKSHEET