Please do not write in the boxes immediately below.

problem	1	2	3	4	5	6	*EC*	total
points								

MATH 244 Linear Algebra Spring 2025 Exam 1

February 26, 2025

Your name____

The exam has 8 different printed sides of exam problems and 1 side workspace.

Duration of the exam is 90 minutes. There are 6 problems, worth 20 points each, and an extra credit problem, worth 5 points. From Problems 1 – 6, only 5 problems will be graded. If you solve all Problems 1 – 6, you must cross out the problem in the box above that must not be graded. If you solve all Problems 1 – 6 and do not cross out a problem, only the first five problems will be graded. Show all your work for full credit. Books, notes etc. are prohibited. Calculators, cellphones, earphones, AirPods and cheat sheets are NOT permitted.

1. Determine whether the statements that follow are true or false. Give a specific counterexample when false and a reason when true.

    (i) $(AB)^2 = A^2B^2$

    (ii) The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.

    (iii) An m by n matrix has no more than m pivot variables.

    (iv) If AB and BA are defined then A and B are square matrices.

    (v) If the coefficient matrix A is not invertible, the system $A\vec{x} = \vec{b}$ always have infinitely many solutions.

(vi) If a diagonal entry of an upper triangular matrix is zero, then the matrix is singular.

(vii) If A and B are matrices of the same size, then the formula $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$ must hold.

(viii) If rows 2 and 4 of A are the same, so are rows 2 and 4 of AB.

(ix) If A and B are skew-symmetric then AB is skew-symmetric.

(x) A 10 by 10 matrix with two identical rows is not invertible.

2. Let k be an arbitrary constant. Consider the equations

$$\begin{aligned} x + 2y + 3z &= 4 \\ x + ky + 4z &= 6 \\ x + 2y + (k+2)z &= 6, \end{aligned}$$

(a) What is the augmented matrix of this system of equations?

(b) For which values of the constant k does this system have a unique solution?

(c) When is there no solution?

(d) When are there infinitely many solutions?

3. (a) Find all vectors in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(b) A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following facts:

(i) All homogeneous systems are consistent.

(ii) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

(iii) If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_2$ is a solution as well.

(iv) If $\vec{x}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and k is an arbitrary constant, then $k\vec{x}$ is a solution as well.

4. For which values of the constants a, b, c, and d is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ a linear combination of $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}$?

5. (a) Find the classical adjoint of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 6 & 6 \end{bmatrix}$ and use the result to find A^{-1}.

(b) Solve the system.

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 0 \\ x + 6y + 6z &= 0 \end{aligned}$$

6. (a) Find the **cosine** of the angle between the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

(b) Calculate the area of the triangle with vertices $P = (0, 0, 0)$, $Q = (1, 3, 2)$, and $R = (1, 5, 7)$.

(c) Compute the volume of the parallelepiped spanned by $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Extra Credit Problem (5 points) List the rrefs in the order of the rank, and denote an entry that can be any real number by $*$.

1. How many types of 2×2 matrices in reduced row-echelon form are there?

2. How many types of 3×2 matrices in reduced row-echelon form are there?

3. How many types of 3×3 matrices in reduced row-echelon form are there?

WORKSHEET