(1) Verify that the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^3$ are orthonormal. Can you find a vector $\mathbf{u}_4 \in \mathbb{R}^4$ such that all the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ are orthonormal?

(2) Consider the subspace $V = \mathrm{im}(A)$ of $\mathbb{R}^4$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Find $\mathrm{proj}_V(\mathbf{x})$ for

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}.$$

(3) Consider the orthogonal projection $T(\mathbf{x}) = \text{proj}_V(\mathbf{x})$ onto a subspace V of $\mathbb{R}^n$. Describe the image and kernel of T.

(4) Consider a subspace V in $\mathbb{R}^n$. The *orthogonal complement* $V^\perp$ of V is the set of those vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in V:

$$V^\perp = \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{v} \cdot \mathbf{x} = 0, \text{ for all } \mathbf{v} \in V\}.$$

Note that $V^\perp$ is the kernel of the orthogonal projection onto V. .

Some properties of the orthogonal complement

Consider a subspace V of $\mathbb{R}^n$. Then

- The orthogonal complement $V^\perp$ of V is a subspace of $\mathbb{R}^n$.
- The intersection of V and $V^\perp$ consists of the zero vector: $V \cap V^\perp = \mathbf{0}$.
- $\dim(V) + \dim(V^\perp) = n$.
- $(V^\perp)^\perp = V$.

Can you prove the properties of the orthogonal complement?

(5) Consider the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^4.$$

Find a basis of the subspace of $\mathbb{R}^4$ consisting of all vectors perpendicular to $\mathbf{v}$.

(6) Find a basis for $W^\perp$, where

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right).$$