Pappus on the “Delian problem” – Doubling the Cube

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Plan for this talk

- Some background on Pappus of Alexandria and his *Mathematical Collection*
- The “Delian problem” and the problem of the two mean proportionals
- Pappus’s discussion of solutions and the method he claims as his own invention
- Pappus is called on to provide an “expert opinion” about another solution and his response

What we know about Pappus of Alexandria

- Dating data: He mentions Claudius Ptolemy (ca 100–170 CE) while Proclus (412–485 CE) mentions him
- A 10th century CE Byzantine encyclopedia, the *Suda*, says he was active at the same time as Theon of Alexandria (best known to us as an editor of Euclid’s *Elements* and as the father of Hypatia), in the reign of the Emperor Theodosius I (372–395 CE)
- Recent scholarship has settled on the somewhat earlier dates ca. 290–350 CE, based on an eclipse described in Pappus’s commentary on Ptolemy’s *Almagest*–matches one known to have occurred in 320 CE.
- He may have been attached to the Museum in Alexandria as a teacher of mathematics

His major work – the *Mathematical Collection*

- Pappus lived near the end of the history Greek mathematics, long after the time when most original discoveries were made
- A wide-ranging “collection” of expositions of earlier mathematicians’ work, and (possibly) some original items
- Modern evaluations of his work vary:
    1. Some (e.g. T.L.Heath) think he was trying to re-stimulate interest in classical geometry and preserve texts that had become rare
    2. Others (e.g. S. Cuomo, R. Netz, A. Jones, W. Knorr) see his commentaries on earlier work as the primary form of scholarship of his time and see him as almost totally unoriginal (and sometimes not even especially competent)

A "collection" of paper drafts or a connected work?

One recent view:

> *The Collection, therefore, appears as a volume of collected works, put together by an editor whom we could describe as a 'literary executor', and who was more concerned with faithfully preserving Pappus's various papers than with creating an intelligible or useful work. (A. Jones, Book 7 of the Collection, 26)*

The *Mathematical Collection*, cont.

- Arranged into 8 (or possibly more) books: Book I is lost; Book II (on arithmetic) survives only in fragmentary form; Books III - VIII (on geometric, astronomical, and mechanical topics) essentially complete (?)
- *Historical importance (a):* We *only* know about many of the texts and mathematical developments Pappus discusses from Pappus himself (and a few other commentaries on other works!)

Transmission of Pappus

- The earliest known surviving manuscript of the *Mathematical Collection* is known as *Vat. gr. 218*; probably made in Constantinople in 10th century CE; other manuscripts derived from this one also known
- But not widely known or even mentioned, either in the Islamic world or in Europe, until 1588 when Federico Commandino’s Latin translation was published
- *Historical importance (b):* Immensely influential on the development of European mathematics in the Renaissance. Many works of Viète, Clavius, Descartes, etc. were directly inspired by reading Commandino’s translation–e.g. Descartes’ approach to geometry via coordinates and algebra.

The three Greek construction problems

- Much of the Greek (pure) mathematics that we know was developed in attempts to solve three famous problems:
- (Duplication of the cube): Given a cube, construct a cube with twice the volume
- (Trisection of a general angle): Given an arbitrary plane angle, construct an angle one third as large
- (Squaring the circle): Given a circle, construct a square with the same area
- What counts as a *construction*, though? Most restrictive sense: Using only the (unmarked) straightedge and ("collapsing") compass from postulates of Euclid.
- However, the Greeks *did not* stop there, as we will see.

Duplication of the cube

- Very early in the history of work on cube duplication, Hippocrates of Chios (ca. 470 – 410 BCE) realized that if *MN* and *KL* are found such that

$$(2 \cdot AB) : KL = KL : MN = MN : AB$$

  then *MN* is the edge of the cube with twice the volume of the cube with edge *AB* (since then by multiplication $(2 \cdot AB) : AB = 2 : 1 = (MN : AB)^3$ and hence $MN^3 = 2 \cdot AB^3$. Algebraically (i.e. anachronistically), $MN/AB = \sqrt[3]{2}$.)
- The *KL* and *MN* were called *mean proportionals in continued proportion* between $2 \cdot AB$ and *AB*.
- The factor of 2 can also be replaced by any other ratio

An interesting methodological passage

The ancients say that there are three sorts of geometric problems–called planar, solid, and line-like. Those that can be solved with lines and circles are reasonably called planar because the lines and the circles by which they are solved have their origin in the plane. Those that are solved through one or more conic sections are called solid because for the construction it is necessary to make use of sections of the surfaces of solid bodies (I speak of the conic surfaces). The third type is called line-like ... (Pappus, Book III, 20; translation JL)

Line-like problems make use of auxiliary curves such as spirals, quadratrices, conchoids, cissoids, etc.

How finding the two mean proportionals fits into the classification

- Pappus says it was (thought to be) a *solid* problem
- For instance, one of the solutions ascribed to Menaechmus (4th century BCE) by Eutocius of Ascalon (ca. 480 - 540 CE) in his commentary on Archimedes' *On the sphere and cylinder* uses a hyperbola and a parabola
- In modern terms, if the hyperbola $xy = 1$ is intersected with the parabola $y^2 = ax$, then the y-coordinate of the point of intersection in the first quadrant satisfies $y^3 = a$
- Can also be rephrased in terms of finding two mean proportionals between 1 and a.

Some comments

- (As far as we know) the Greeks never actually *proved* that the problem was not planar
- But coordinates and algebra made it possible to do that–Descartes had most of the ideas for a proof, and this was finally nailed down by Pierre Wantzel in the early 19th century
- *Pappus continues:* But since it was difficult to draw conic sections in the plane, solutions using instruments or limiting constructions were developed
- He gives accounts of methods due to Eratosthenes (3rd century BCE, using the *mesolabe*), Nicomedes (3rd century BC), and Heron (1st century CE)

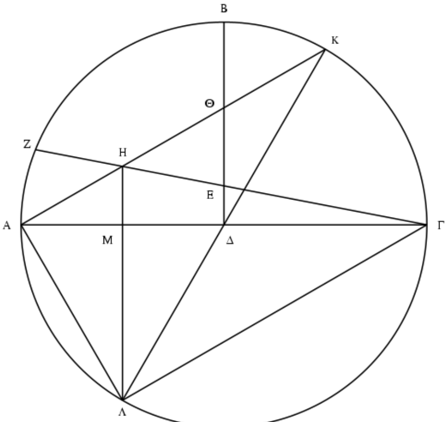
Pappus’s own solution

He then adds a solution he claimed to have found himself (he was proud enough of the following that he included it twice, once in Book III and again in Book VIII)

Theorem

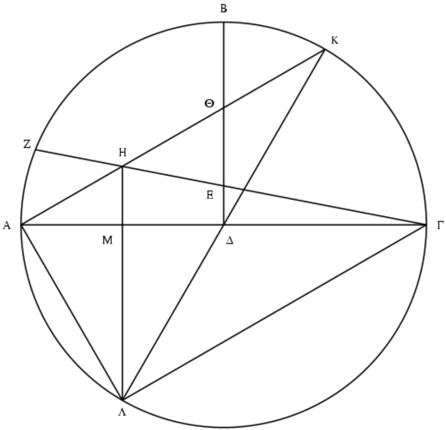
Not only the doubled cube, but a cube that has any given ratio to a given cube, can be found using instruments.

Pappus's solution



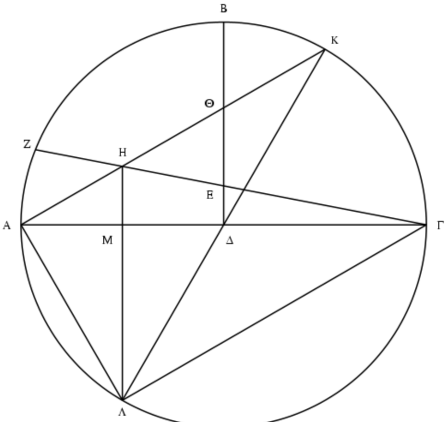
- Let a circle ΑΒΓ with diameter ΑΓ have been constructed, and let a perpendicular ΒΔ be erected from the center Δ.
- Let a ruler (i.e. the line ΑΚ) be moved about the point Α, "as though that end was fixed at Α by a small nail," while the other end moves between Γ and Β.

The construction



- Let the desired ratio between the two cubes be the ratio $\mathrm{B}\Delta : \Delta\mathrm{E}$.
- Join $\Gamma\mathrm{E}$ and produce it to Z on the circle.
- Let $\mathrm{H} = \mathrm{AK} \cap \Gamma\mathrm{Z}$ and $\Theta = \mathrm{AK} \cap \mathrm{B}\Delta$
- Turn the ruler until $\mathrm{H}\Theta = \Theta\mathrm{K}$ (this must happen by continuity)
- Then I claim $(\mathrm{B}\Delta)^3 : (\Delta\Theta)^3 = \mathrm{B}\Delta : \Delta\mathrm{E}$.

The proof of the claim



- Join $\mathrm{K\Delta}$ and produce to Λ on the circle. Then $\Lambda\mathrm{H} \parallel \mathrm{B\Delta}$ and $\mathrm{HM} \perp \mathrm{MA}$.
- $\Gamma\mathrm{M} : \mathrm{MA} = \mathrm{AM}^2 : \mathrm{MH}^2$ since triangles AMH, $\Lambda\mathrm{MA}$, and $\mathrm{A\Delta\Theta}$ are all similar.
- Multiply by the ratio $\mathrm{AM} : \mathrm{MH}$ so $\Gamma\mathrm{M} : \mathrm{MH} = \mathrm{AM}^3 : \mathrm{MH}^3$.
- But $\Gamma\mathrm{M} : \mathrm{MH} = \Gamma\Delta : \Delta\mathrm{E} = \mathrm{B\Delta} : \Delta\mathrm{E}$ and $\mathrm{AM} : \mathrm{MH} = \mathrm{A\Delta} : \Delta\Theta = \mathrm{B\Delta} : \Delta\Theta$. □

Comments on the construction

- If $\mathrm{B\Delta} : \Delta\Theta = \Delta\Theta : \Delta\mathrm{N}$, then $\Theta\Delta$ and $\Delta\mathrm{N}$ are two mean proportionals between $\mathrm{B\Delta}$ and $\Delta\mathrm{E}$
- Finding $\mathrm{H}\Theta = \Theta\mathrm{K}$ is done essentially by *trial and error:*
- Moving ruler $\Rightarrow$ an "instrumental" construction–*neusis* (i.e. "verging," or, in our terms, *limiting*)
- Some input from the senses of the geometer is needed for the segments to be judged as equal
- That will almost never be attained exactly; in modern terms, this is an *approximate solution*
- The rest of the construction uses only lines and circles; but this is *not a planar construction* according to Pappus's classification because of this

What does this have to do with mechanics?

- Recall that I mentioned before that, in addition to the place in Book III, Pappus includes another almost word-for-word copy of this result in his Book VIII devoted to *mechanics*
- What's the connection?
- This and other solutions of the Delian problem give ways to scale solid objects up or down in a given ratio–at least potentially useful in design of weapons, architecture, etc.
- (Maybe counterintuitively for us) for the Greeks, this was (at least in part) an *applied problem*!
- *In fact*, Heron's discussion of his own solution actually comes from his treatise on the construction of *catapults*

Pappus’s introductory remarks

- Before the sections of Book III discussed above, Pappus begins with an interesting introduction
- Like several of the other books, this is addressed to a certain person–here someone named *Pandrosion* (πανδροσίων), usually taken to be a fellow Alexandrian mathematical contemporary
- *Comment:* Some modern authors discussing this section say Book III is “dedicated to” Pandrosion, but that is *definitely not correct* as we will see
- The mentions of Pandrosion by Pappus are the only surviving record of that person, whoever they were

The beginning of Book III

Those wishing to distinguish more skillfully between the results sought in geometry, most excellent Pandrosion, think it fit to call that in which it is proposed to do or to construct something a 'problem,' and that in which, certain things having been assumed, one observes what follows from them and everything that goes along with them a 'theorem.' (Book III, 1, translation JL)

Some corollaries, according to Pappus

- Someone who poses a problem is hardly responsible if what is proposed to be done or constructed turns out to be impossible
- Indeed, part of solving a problem is determining whether the thing proposed is possible or not; if so, how, and in how many ways it is possible
- On the other hand, someone stating a theorem is to be held responsible if they turn out not to have carefully considered the consequences of their starting assumptions or if they present an incorrect or incomplete proof

An interesting sidelight on this passage

- A standard scholarly edition of the whole *Mathematical Collection* – one of the earliest such editions still in use – was edited by Friedrich Hultsch, and published in 3 volumes, 1876-1878
- Hultsch examined at least three of the surviving manuscripts to assemble his Greek text, added an *apparatus criticus* showing variant readings, a Latin translation (drawing on, but better than, Commandino's Latin version), and extensive commentaries
- However, in this particular passage, he actually did something impossible to justify

Multiple errors

- Commandino mistook the Greek word κρατίστη (what I translated as "most excellent") for a name, writing "Cratiste" in Latin (Note: no capitalization in original)
- In his *apparatus*, Hultsch says that Commandino's reading must be corrected, then opts to overrule three of the manuscripts he consulted, reads κράτιστε, instead of κρατίστη
- What's in a vowel? The form κρατίστη is the *feminine vocative form*, while κράτιστε is the *masculine vocative*
- In his index, Hultsch acknowledges some scholars say the -η indicates a female name, but he prefers the masculine because the (nominative) -ιων ending in πανδροσίων appears in other masculine names (!!)

In current scholarship, ...

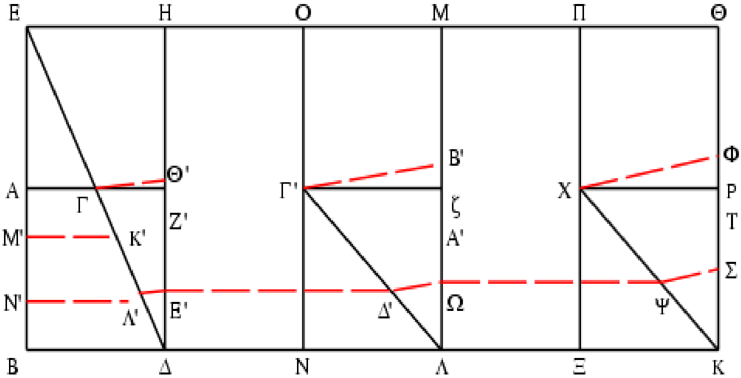
- It's generally accepted that Pandrosion was indeed a woman and Hultsch has been duly chastised for his apparent prejudice
- The name itself is actually a diminutive form of the name of one of the daughters of Kekrops (the legendary first king of Athens) meaning "all-dewy" – not likely as a man's name(!)
- If our dates for Pappus are correct, Pandrosion lived two or three generations before Hypatia–a new contender for the *earliest female mathematician for whom we have evidence*
- But references to Pandrosion as male still crop up in various places

Back to beginning of Book III

Pappus continues, still addressing Pandrosion:

> *Recently some people professing to know mathematics through you [I read this as: as a result of your teaching] set down the enunciations of some problems ignorantly (ἀμάθως). ... They claimed to know that two mean proportionals between two lines could be found by a planar construction and they thought that I was the man to judge their construction. This is the way they said to do it: (Book III, 1, translation JL)*

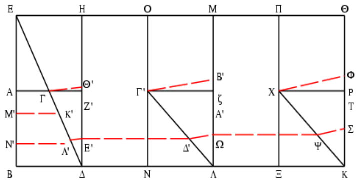
The figure



The proposed construction, step by step

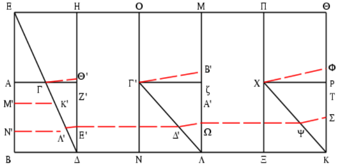
- Let AB and $\mathrm{A\Gamma}$ be the two lines, placed at right angles at A
- Take $\mathrm{AB} = \mathrm{B\Delta} = \mathrm{\Delta N} = \mathrm{N\Lambda} = \mathrm{\Lambda\Xi} = \mathrm{\Xi K}$.
- Extend $\Delta\Gamma$ to E on BA, produced
- Take $\mathrm{E\Theta}$ parallel to BK and connect $\mathrm{H\Delta}, \mathrm{ON}, \mathrm{M\Lambda}, \Pi\Xi, \Theta\mathrm{K}$ perpendicular to $\mathrm{E\Theta}$ and BK.
- Let $\mathrm{PK} = \mathrm{AB}$ and Σ be the the midpoint of PK
- Find Φ, T so that $\mathrm{K\Theta} : \Theta\Sigma = \Theta\Sigma : \Theta\mathrm{T} = \Theta\mathrm{T} : \Theta\Phi$
- Note: This means that $\Theta\Sigma$ and $\Theta\mathrm{T}$ are two mean proportionals between $\mathrm{K\Theta}$ and $\Theta\Phi$; this can be done by finding 4th proportionals one step at a time since K, Θ, Σ are all determined by given information

The proposed construction, step by step, continued



- Connect ΦX and draw $\Sigma\Psi$ parallel to it, with Ψ on XK
- Take $\Omega\Psi$ parallel to BK, with Ω on ΛM
- Find B' and Σ' so that $\Lambda M : M\Omega = M\Omega : MA' = MA' : MB'$
- Join $B'\Gamma'$ and make $\Omega\Delta'$ parallel to it with Δ' on $\Gamma'\Lambda$

The proposed construction, step by step, concluded



- Take a horizontal line through Δ', meeting $\Delta\mathrm{H}$ in E'
- Make $\Delta\mathrm{H} : \mathrm{HE}' = \mathrm{HE}' : \mathrm{HZ}' = \mathrm{HZ}' : \mathrm{H}\Theta'$
- Connect $\Theta'\Gamma$ and draw $\mathrm{Z}'\mathrm{K}'$, $\mathrm{E}'\Lambda'$ parallel to it
- Take horizontals through K' and Λ' over to AB. Claim: $\mathrm{M}'\mathrm{K}'$ and $\mathrm{N}'\Lambda'$ are the mean proportionals

Pappus's "referee's report"

Since the philosopher Hierios and many of his companions who are students of mine have thought it right for me to render a judgment on this construction, while "the other guy" has only promised to give a proof for it,[1] *I have to say now that it is not correct. The construction was enunciated inexpertly (ἀπείρως). (Book III, 3, translation JL)*

[1] Apparently, the proposer did not supply a proof; he just claimed that this would work.

A comment

The quotations I have translated from Book III probably sound somewhat pedantic and obnoxious to our modern ears. It is hard (for me, at least) to judge how they would have come across to Alexandrians of Pappus’s time. Greek culture, through its whole history, thrived on combative debates in all sorts of contexts (see article by A. Bernard cited below, which tries to put Pappus into the milieu of Greek rhetoricians and sophists).

Pappus’s criticisms

- Main criticism: this is not a planar problem, so the proposed construction cannot be (exactly) correct. (Pappus is right about that!)
- Pappus shows that the proposed figure is not always correct either–the point T will lie above P in some cases and below P in others
- There is exactly one value of the ratio $\rho = \mathrm{EB} : \mathrm{AB} = \Theta\mathrm{K} : \mathrm{KP}$ for which the first step finds the two mean proportionals exactly, but in other cases, even the three steps don’t give an exact solution
- Pappus psychologizes: He thinks the proposer realized that using mean proportionals is circular, but essentially tricked himself into believing that several steps would fix that.

Modern interpretations

- With the advantages of algebra and analysis developed after Pappus’s time, in 1873, R. Pendlebury pointed out that the proposed solution gives an iteration that *converges to* the correct mean proportionals if repeated indefinitely
- Discussed in T.L. Heath’s history of Greek mathematics
- Wilbur Knorr said he thought Pappus had (in effect) “missed the point” with his criticism because he didn’t understand that this was an iteration
- But isn’t that anachronistic? (My take: That’s not what the proposer claimed and the process is not framed as an exhaustion argument as in Euclid or Archimedes.)
- Final irony: even *Pappus’s own method* would need something like that to produce exact mean proportionals(!)

Thanks for your attention!

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