MATH 361 Real Analysis **Homework Assignment 9** **Spring 2024**

Due by 4pm on April 19. Do not forget to attach the honor code.

1. (10 points) Prove the following:
   (a) Every real number can be expressed as a decimal expansion, but not uniquely.
   (b) Every real number can be expressed uniquely as a decimal expansion that does not end with an infinite string of 9s.

2. (10 points) Decide whether each of the following series converges or diverges:
   (a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$
   (b) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

3. (10 points) Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.
   (a) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.
   (b) If $\sum a_n$ converges conditionally, then If $\sum n^2 a_n$ diverges.

4. (10 points) Give an example of each or explain why the request is impossible referencing the proper theorem(s).
   (a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.
   (b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

5. (5 points each) Let $A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots \right\}$.
   (a) What are the limit points?
   (b) Is the set open? Closed?
   (c) Does the set contain any isolated points?
   (d) Find the closure of the set.

6. (5 points each) Let $B = \{x \in \mathbb{Q} : 0 < x < 1\}$.
   (a) What are the limit points?
   (b) Is the set open? Closed?
   (c) Does the set contain any isolated points?
   (d) Find the closure of the set.

7. (5 points each) Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.
   (a) $\mathbb{Q}$
   (b) $\mathbb{N}$
   (c) $\{x \in \mathbb{R} : x \neq 0\}$
   (d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$